A novel quantum-mechanical interpretation of the Dirac equation

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Abstract

A novel interpretation is given of Dirac's "wave equation for the relativistic electron" as a quantum-mechanical one-particle equation. In this interpretation the electron and the positron are merely the two different "topological spin" states of a single more fundamental particle, not distinct particles in their own right. The new interpretation is backed up by the existence of such "bi-particle" structures in general relativity, in particular the ring singularity present in any spacelike section of the spacetime singularity of the maximal-analytically extended, topologically non-trivial, electromagnetic Kerr-Newman spacetime in the zero-gravity limit (here, "zerogravity" means the limit $G \to 0$, where G is Newton's constant of universal gravitation). This novel interpretation resolves the dilemma that Dirac's wave equation seems to be capable of describing both the electron and the positron in "external" fields in many relevant situations, while the bi-spinorial wave function has only a single position variable in its argument, not two as it should if it were a quantum-mechanical two-particle wave equation. A Dirac equation is formulated for such a ring-like bi-particle which interacts with a static point charge located elsewhere in the topologically non-trivial physical space associated with the moving ring particle, the motion being governed by a de Broglie-Bohm type law extracted from the Dirac equation. As an application, the pertinent general-relativistic zero-gravity Hydrogen problem is studied in the usual Born–Oppenheimer approximation. Its spectral results suggest that the zero-GKerr-Newman magnetic moment be identified with the so-called "anomalous magnetic moment of the physical electron," not with the Bohr magneton, so that the ring radius is only a tiny fraction of the electron's reduced Compton wavelength.

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1 Introduction

We begin with a brief history of Dirac's equation and its quantum mechanical interpretations. Readers who are familiar with this subject may wish to skip ahead to section 1.2 for an introduction to our novel proposal, and for an executive summary of our main results at the end of that section.

1.1 Quest for the quantum-mechanical interpretation of Dirac's equation

Many textbooks and monographs (e.g. [60], [40], [81], [70]) tell the story of Dirac's marvelous discovery of the special-relativistic generalization of Pauli's non-relativistic spinor wave equation for the "spinning electron": his ingenious insight that a first-order partial differential operator is needed rather than the second-order wave operator in the so-called¹ Klein–Gordon equation; his equally ingenious insight that complex four-component bi-spinors, instead of Pauli's complex two-component spinors, are needed to accomplish this goal; his consequential formulation of the equation and his skillful analysis of the same; the g-factor 2; the explicit solution of the Hydrogen problem in terms of elementary functions (by Darwin [24] and Gordon [39]), and the surprising exact agreement of the Dirac point spectrum with Sommerfeld's fine structure formula (save the labeling of the energy and angular momentum levels);² the strange occurrence of a negative energy continuum below $-mc^2$ (with m the electron's rest mass, and c the speed of light in vacuum), leading to Dirac's ultimate triumph: the prediction of the existence of the anti-electron (a.k.a. positron) — based on his "holes in the Dirac sea" interpretation.

Yet, not all is well. The most perplexing (and intriguing) part of the above success story is that it is based on Dirac's changing the rules of the game while he was playing it. Thus, originally formulating it as a quantum-mechanical one-particle wave equation on one-particle configuration space, Dirac — by postulating that all negative energy continuum states are occupied with electrons — switched to an infinitely-many particle interpretation on physical space: a "poor man's quantum field theory" (yet an important stepping stone towards quantum electrodynamics, one that still is the subject of serious studies by mathematical physicists [58]). And while it is difficult to argue with practical success, Dirac's ad-hoc "holes in the negative energy sea" theory can be criticized for de-facto bypassing the conceptual problem of the proper quantum-mechanical interpretation of Dirac's equation (assuming there exists one!), rather than addressing it.

Stückelberg [73] and Feynman [34] revived the quantum-mechanical interpretation of Dirac's equation and argued that it is an equation for both, the electron and the positron, cf. Thaller [80]: "[U]p to now no particular quantum mechanical interpretation [of Dirac's wave equation] is generally accepted. ... [T]he Stückelberg–Feynman interpretation (Stückelberg 1942, Feynman 1949) ... is intermediate between a one-particle theory and Dirac's hole theory, because it claims that the Dirac equation is able to describe two kinds of particles, namely electrons and positrons (but not their interaction; negative energy states are directly observed as positrons with positive energy)." Thaller, who adopts the Stückelberg–Feynman interpretation in his scattering theory, goes on to emphasize that it is "formulated in the language of wave packets ... and does not rely on unobservable objects like the Dirac sea." In a nutshell, the main idea is that wave packets composed of only positive energy eigenfunctions and scattering states "describe" positrons; cf. also [16]. The problem with identifying electrons and positrons with respect to the respective subspaces of the "free" Hamiltonian is that switching on "external" fields may induce transitions between

¹We are alluding to the history of this equation: first contemplated by Schrödinger *before* he discovered his nonrelativistic equation, and rediscovered by various physicists, amongst them Klein, Gordon, and also Pauli.

 $^{^{2}}$ For a modernized semi-classical Bohr–Sommerfeld approach that leads to the correct labeling, see [49].

the two; or, in mathematical language: the positive and negative energy Hilbert subspaces for the free Hamiltonian may not be invariant under the unitary evolution of a scattering problem. Klein's paradox (see [81]) shows that these subspaces are not always invariant under the unitary evolution, indeed. Furthermore there are examples of physically interesting electric potentials (e.g. a regularized Coulomb potential) which can generate negative energy eigenstates which clearly are to be interpreted as belonging to the electron, not the positron; see [40]. Even in favorable situations, mixed initial conditions can pose an interpretational dilemma.

Thus not all is well with the Stückelberg–Feynman interpretation either. To Thaller's emphasis of its limitations we here add another troublesome criticism, namely that this interpretation is in conflict with established quantum-mechanical many-body concepts. Thus, while Dirac's bispinorial wave function depends (beside time) only on one position variable, it should have two position variables in its arguments, not merely one, if Dirac's equation truly were a quantummechanical two-particle equation. For then it has to reproduce Pauli's quantum-mechanical twoparticle equation for a non-relativistic electron–positron pair in the non-relativistic limit, and this Pauli equation — which is the traditional starting point for computing the leading-order spectral properties of positronium, with relativistic corrections computed perturbatively subsequently [4] does have a complex four-component wave function with two position variables in its arguments.³

1.2 A novel proposal

Since the enigmatic quantum-mechanical character of "Dirac's equation for the electron" manifests itself in the seemingly contradictory aspects that, on one hand, it has all the formal hallmarks of a *single-particle* equation, while, on the other, its set of solutions covers (much of) the physics of *both* the *electron* and the *anti-electron*, the only logical conclusion that avoids this paradox is that electron and anti-electron are not separate entities but merely "two different sides of the same medal." By this we do not mean the well-known charge-conjugation transformation which maps a "negative-energy Dirac electron state" into a "positive-energy Dirac positron state," and vice versa; neither do we mean the interpretation suggested by Stückelberg, Wheeler, and Feynman that an "electron moving backward in time appears as a positron moving forward in time." Rather, we propose that *electron and anti-electron are two different "topological-spin" states of a single, more structured particle* (moving forward in time).

Our novel interpretation is backed up by the fact that such bi-particle structures exist in some electromagnetic spacetime solutions of Einstein's classical theory of general relativity.⁴ More concretely, we mean the structure commonly known as the "ring⁵ singularity" of the maximal-

³There is one loose end in this argument — which can easily be tied up. More to the point, as in the quantummechanical analog of the Kepler problem, one can change coordinates from the two particle position vectors (defined w.r.t. some inert fixed point) to center-of-mass coordinates (w.r.t. the same fixed point) plus relative coordinates. In the spinless non-relativistic limit the center-of-mass coordinates can be factored out, and the remaining wave equation in the relative coordinates is effectively a one-body equation for a test particle in an external Coulomb field. So one may speculate whether Dirac's equation is perhaps of this type, describing the *relative* motion of electron and positron in relative coordinates. Unfortunately the mass parameter m in Dirac's equation "for the relativistic electron" is a factor 2 too big for this interpretation to be feasible, putting to rest speculations in this direction.

⁴By invoking a classical physical theory, in our case Einstein's general relativity, we are simply following the standard protocol of "first quantization." Enjoying the benefit of hindsight we may simplify our task and look for the desired electromagnetic "bi-particle structure" in classical physical theory, then compute its electromagnetic interaction with some other classical electromagnetic object, in particular a point charge. This classical interaction then enters the pertinent quantum-mechanical wave equation obtained from "first quantization," in our case Dirac's equation. The remaining task then is to show that this Dirac equation makes the right physical predictions and avoids any of the paradoxical aspects that we have discussed.

⁵What is "ring-like" is actually a constant-t "snapshot" of the spacetime singularity.

analytically extended, topologically non-trivial,⁶ electromagnetic spacetime of Newman et al. [64, 14] — in its zero-gravity limit. Here, "zero-gravity limit" means the limit $G \rightarrow 0$, where G is Newton's constant of universal gravitation, applied to the Kerr–Newman metric⁷ expressed in any one of the most symmetric global coordinate charts.

1.2.1 The zero-gravity Kerr–Newman (zGKN) spacetime.

As shown rigorously in an accompanying paper [77], this zGKN spacetime itself is a flat but topologically nontrivial solution of the vacuum Einstein equations $R_{\mu\nu} = 0$, while its electromagnetic fields are static Appell–Sommerfeld solutions⁸ of the linear Maxwell equations on this spacetime, with well-behaved sesqui-pole sources concentrated on its spacelike ring singularity. Most importantly, the ring singularity of a spacelike snapshot of the spacetime appears to be positively charged in one "sheet," yet negatively charged in the other, with pertinent electric monopole asymptotics near spacelike infinity (see Fig. 1); analogous statements hold for the magnetic structure, with dipole asymptotics near spacelike infinity. The *electromagnetically anti-symmetric spacelike ring singularity of this double-sheeted Sommerfeld space* is precisely the kind of bi-particle-like structure which vindicates our informal wording that "electron and anti-electron are two sides of one and the same medal" as mathematically realized in a classical physical field theory, and therefore as putatively physically viable in a quantized theory.

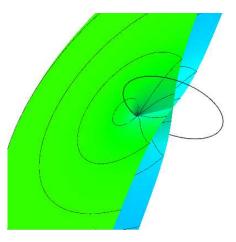


Figure 1: An illustration of the zGKN spacetime. Shown are the ring singularity and one of the constantazimuth sections of a constant-time zGKN snapshot immersed into three-dimensional Euclidean space; the two sheets of this flat Sommerfeld space are slightly bent for the purpose of visualization. The apparent intersection of the two sheets along a radius of the ring singularity is a consequence of the immersion and not a faithful representation. Also shown are equipotential lines of the Appell–Sommerfeld electric field; the ring singularity appears positively charged in one sheet, and negatively in the other, so that far from the ring the electric field becomes positive, respectively negative monopole-like, whereas inside a ball having the ring singularity as equator it appears dipolar, with field lines transiting from one sheet to the other.

⁶The complement of a wedding ring in three-dimensional Euclidean space is topologically non-trivial, too, but "looping through the ring once brings you back to where you began;" in a spacelike slice of the maximal-analytically extended Kerr–Newman spacetime "you need to loop through the ring twice to get back to square one."

⁷A priori speaking the $G \rightarrow 0$ limit should also be applied to the expressions for the Kerr–Newman electromagnetic fields, but these are *G*-independent in the coordinates we use.

⁸Remarkably, the electromagnetic fields were discovered in 1887 by Appell [1] by performing a complex translation "of physical space" in the Coulomb potential. However, Appell did not realize, apparently, that the complex generalization of the Coulomb potential lives on a topologically non-trivial extension of Euclidean \mathbb{R}^3 . This insight is due to Sommerfeld [71], who is credited with introducing the concept of *branched Riemann space*, a three-dimensional analog of a Riemann surface, on which multi-valued harmonic functions such as the Appell fields become single-valued.

We stress the importance of the zero-gravity limit of the maximal-analytically extended Kerr-Newman spacetime and its electromagnetic fields [77] for getting our electron/anti-electron biparticle interpretation of its ring singularity off the ground. The reason is that the energymomentum-stress tensor of the electromagnetic Appell-Sommerfeld fields is not locally integrable at the ring singularity, so when coupled to spacetime curvature by "switching on gravity," it produces the physically "pathological" effects of the G > 0 maximal-analytically extended Kerr-Newman spacetime (such as closed timelike loops; the t = const. ring singularity itself is also timelike when G > 0 and not interpretable as a stationary particle-like source [19])⁹ — more on this in section 2. The $G \rightarrow 0$ limit removes all these troublesome features of the Kerr-Newman spacetime while retaining the topological and electromagnetic features that can be given a clear physical meaning.¹⁰ **1.2.2 The physical spacetime of a moving zGKN-type ring singularity**.

Having identified a candidate structure for our electron/anti-electron bi-particle proposal in form of the ring singularity of a snapshot of the static zGKN spacetime, we next describe the *physical spacetime* associated with such a ring-like bi-particle when moving, were motion is understood as relative to an inertial frame attached to an arbitrarily chosen origin (the analog of Carl Neumann's " α body frame" for Newton's mechanics). We restrict our considerations mostly to the *quasi-static motions* approximation and only briefly comment on more general motions.

In the simplest special case, when the ring particle is at rest relative to the α frame, the spacetime is simply (a spacelike translation and rotation of) the static zGKN spacetime, of course. Furthermore, a Lorentz boost of such a static zGKN spacetime coordinates produces the spacetime of a ring singularity in straight inert motion relative to the α frame; see Fig. 2 for a totally stripped

¹⁰It is our pleasant duty to note here that we are not the first to try to associate the double sheetedness of the zGKN spacetime and its anti-symmetric electromagnetism with particle / anti-particle symmetry. Namely, after reading an earlier version of our paper, Prof. Ted Newman kindly send us a letter with (amongst other valuable information) the following historical note: "I had (in about 1966) myself noted — in the G = 0 limit — that by going along the axis of symmetry from large r down to zero and then to negative r that the sign of the charge would change. Kerr and Penrose did clarify this for me. I and one of my Polish colleagues, spent a great deal of time — talking and trying out ideas concerning the double sheetedness — mainly trying to put it into the context of parity, time reversal and charge conjugation symmetry. It was a very enticing idea. Unfortunately we never could make it into a unified theory or even a coherent idea. We eventually gave it up — not because it was wrong — but because we were stuck and could not make it work." In this paper we will see that the double sheetedness and its charge (anti-)symmetry are associated not with parity, time reversal, and charge conjugation symmetry operations, but with a novel symmetry operation on the Dirac bi-spinor, associated with what we call "topo(logical)-spin" of the zGKN ring singularity, viz. the particle/anti-particle symmetry of the zGKN ring singularity is implemented in a topo-spin flip operation and, thus, is somewhat akin to Heisenberg's iso-spin concept for nucleons.

⁹While "switching off gravity" should not be viewed as a troublesome step, our statement that the zero-G limit is important to get our electron/anti-electron bi-particle interpretation of its ring singularity off the ground, in the sense that the KN "ring singularity" cannot be given a bi-particle interpretation for G > 0, could seem to deal a fatal blow to our proposal. However, the G > 0 problems we have in mind are similar to those affecting the usual textbook formulation of Dirac Hydrogen, viz. "Dirac's equation for a point (test) electron in the field of a point proton at rest in Minkowski spacetime" — as soon as one "switches on G" to compute the general-relativistic corrections to the Sommerfeld spectrum from eigenstates for Dirac's equation of a point (test) electron in the Reissner-Nordström spacetime, "all hell breaks lose" (see [54] for a review). The reason is once again the local non-integrability of the energy-momentum-stress tensor of the classical electromagnetic field, which in the usual textbook case is the field of a point charge. This clearly suggests that the culprit is Maxwell's linear vacuum field equations, and that replacing them with non-linear field equations that produce an integrable energy-momentum-stress tensor, the G > 0pathologies might go away and general-relativistic gravitational curvature corrections to the spectral features of special-relativistic Dirac Hydrogen may eventually come out not larger than the tiny Newtonian gravitational effects relative to the electromagnetic effects on the non-relativistic Schrödinger spectrum of Hydrogen. More about this in our concluding section. In short, the pathological problems of the G > 0 Kerr-Newman singularity are bad news for the physical interpretation of the maximal-analytically extended Kerr-Newman spacetime with G > 0, but not bad news for our proposal that electron and anti-electron might be the two electromagnetic "sides" of a bi-particle-like ring singularity associated with a two-sheeted space.

down illustration of the physical spacetime with a ring singularity in straight motion contrasted with the corresponding illustration of a test particle in straight motion in the zGKN spacetime.

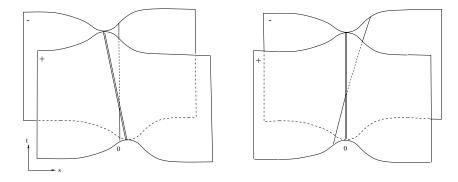


Figure 2: Left: The physical spacetime with a single ring singularity in straight line motion relative to the rest frame of the α point (marked 0), which in the depicted scenario gets "swept over" by the ring singularity; Right: Trajectory of a test particle in straight line motion in the static zGKN spacetime having the center of its ring singularity as the origin (marked 0). The test particle transits through the ring from one sheet to the other. (In both illustrations, the two flat sheets of the double-sheeted spacetimes are bent apart for the purpose of visualization.)

Note that the ring will be circular generally only in its rest frame, whereas a Lorentz boost will render it generally elliptical. However, neglecting effects of order v^2/c^2 as relatively small in the quasi-static motions approximation, a moving ring can be described as circular to leading order.

The physical spacetime with a single moving space-like ring singularity is generally not obtained by a Lorentz boost of zGKN, but in the quasi-static approximation it is foliated by constanttime Sommerfeld spaces which are all snapshots of the zGKN spacetime with a ring singularity translated and rotated relative to the fixed α frame. Unfortunately, it's not so easy to generalize the left illustration in Fig. 2 to visualize a double sheeted spacetime with a ring singularity not moving in a straight line motion with respect to an α point having a vertical world line. Yet, we hope that the general idea of what this physical spacetime is has become clear.

Incidentally, the double-sheeted physical zero-G spacetime with a generally moving ring-like singularity will feature snapshots having deformed ring singularities, and a proper treatment would presumably require solving the so-called vacuum Einstein equations. It is likely that the spacetime will then also feature "gravitational" waves.¹¹

1.2.3 The configuration space(time) of a moving zGKN-type ring singularity.

Finally, to formulate Dirac's equation for the spinor wave function of such a ring-like bi-particle we need to identify the *configuration space of a single ring particle* on which the wave function is defined (as bispinor-valued L^2 function), and this is simply the space of all possible cofigurations for the particle relative to the α frame. In case of the ring singularity, classically knowing its configuration means knowing the location of the center of the ring as well its orientation, i.e. the unit normal to the plane of the ring. Quantum mechanically however, this orientation vector is itself determined by a bi-spinor wave function (as we will show), so that specifying the location of the center of the ring, with respect to a frame attached to the α point of the space, should suffice. It follows that the configuration space for a ring-like particle should also be double-sheeted, since one can contemplate the origin to be "swallowed" by the ring as its center moves, causing the origin to end up on a different sheet of the physical space. Since that is a different configuration

¹¹Note that G = 0 uncouples the spacetime structure from its matter and field content, but the spacetime metric does not need to be flat. In particular, it can feature wave-like disturbances propagating at the speed of "light."

for the ring, one sheet will not be enough to encode all the possible positions of the center of the ring; thus the configuration space is also double-sheeted, and in fact isomorphic to the Sommerfeld space obtained by a constant-time snapshot of the zGKN spacetime as one can easily show. We remark that *formally* the Dirac equation for a quasi-statically moving ring singularity is therefore a bispinor-valued wave equation on the configuration spacetime which is isomorphic to zGKN. It is important not to confuse this configuration spacetime with a physical zGKN spacetime, though.

1.2.4 Enter Dirac's equation.

To test our bi-particle interpretation of the ring singularity of the constant-time snapshot of the maximal-analytically extended zGKN spacetime as an electron/anti-electron structure whose motion relative to an α frame is governed by Dirac's equation we study the pertinent general-relativistic zero-gravity Hydrogen problem in the usual Born–Oppenheimer approximation. We first show that the classical electromagnetic interaction of a static zGKN ring singularity with a point charge located at the α point in the static spacelike slice of zGKN is given by the minimal coupling formula used to describe the interaction of a test point charge in the field of a given zGKN singularity — to avoid confusion with the "naive minimal coupling" evaluation of the Coulomb field of the point charge at the center of the zGKN ring singularity (times its charge parameter), we will call our interaction formula a "minimal re-coupling" term. Next we show that although the zGKNsingularity is not point- but ring-like, in the quasi-static motions regime one and the same Dirac equation covers these two interpretations of the "point charge plus zGKN ring singularity" system (i.e., only the narrative changes); in particular, the eigenstates are captured correctly. In another accompanying paper [53] we have rigorously studied this Dirac equation (in the interpretation "test point charge in the zGKN spacetime"), and in the present paper the results of [53] will be explained in the novel interpretation put forward here. In addition, here we will argue — compellingly as we believe — that our spectral results suggest as choice for the ring diameter of the electron/anti-electron bi-particle structure *not* the reduced Compton wavelength of the electron, \hbar/mc , as proposed in earlier, different studies which aimed at linking the Kerr–Newman spacetime with Dirac's electron (see Appendix C), but only a tiny fraction of it.

Lastly, we will show that with the help of the Dirac bi-spinors de Broglie–Bohm type laws of motion and orientation can be formulated for the zGKN ring singularity in the quasi-static regime. 1.2.5 Summary.

In this paper we propose a fundamentally new quantum-mechanical interpretation of Dirac's equation as a single-particle equation for a fermionic elementary particle which is both an electron and an anti-electron. We

- argue that the primitive ontology of such a bi-particle which is both an electron and a positron is realized by the general-relativistic electromagnetic ring singularity of a constanttime snapshot of the double-sheeted zero-gravity Kerr-Newman (zGKN) spacetime;
- explain that the *physical spacetime* of such a ring-like bi-particle in quasi-static motion with respect to a rest frame attached to an arbitrarily chosen origin (Neumann's α point) is a two-sheeted spacetime with a "wiggly" timelike tubular region traced out by the moving ring singularity, such that every constant-time snapshot (w.r.t. the α frame) is a combined translation and rotation of the constant-time snapshot of the zGKN spacetime;
- explain that the *configuration spacetime* of this ring-like bi-particle in quasi-static motion is isomorphic to the zGKN spacetime — any point in the pertinent configuration space represents a possible position of the *center* of the ring singularity, relative to the α frame;
- show that a sheet swap in configuration space is associated with a *topological spin* operator acting on the bi-spinorial wave function defined on configuration space;

- show how the bi-spinorial wave function on configuration space naturally defines a three-frame attached to the center of the ring singularity, and thereby an orientation (spin) vector;
- formulate quantum laws of motion for the center and orientation of this two-faced particle using the de Broglie–Bohm approach relative to the α frame, involving a bi-spinorial wave function satisfying Dirac's equation on the configurational zGKN spacetime;
- compute the classical electromagnetic interaction in physical space of a quasi-static ring singularity with an infinitely massive point charge (modeling a proton) located at the α point, showing it is identical to the minimal coupling formula of a point charge in the electromagnetic Appell–Sommerfeld (viz. zGKN) fields;
- explain that the Dirac spectrum in our model coincides with the spectrum of a "Dirac point positron" in a given, fixed background zGKN spacetime we then invoke the results we have previously obtained in this test-charge situation, namely: the self-adjoineness of the Dirac Hamiltonian, the symmetry of its spectrum about 0, the essential spectrum being $\mathbb{R} \setminus [-m, m]$, and the existence of discrete spectrum under suitable smallness assumptions;
- argue that the discrete spectrum reproduces the Sommerfeld fine structure formula (in fact, a positive and a negative copy of it) in the limit of vanishing ring radius, and that the magnitude of effects like hyperfine structure and Lamb(-like) shift of the spectral lines put a limit on the size of the ring singularity which corresponds to the identification of the zGKN magnetic moment with the anomalous magnetic moment of the electron.

1.3 The structure of the ensuing sections

In section 2 we summarize the basic formulas of the zGKN spacetimes and their electromagnetic fields, and also some straightforward generalizations of the latter. This material is taken from [77].

Section 3: We formulate the Dirac equation for a zGKN-type ring singularity that interacts with an infinitely massive static point charge located at the α point in this topologically nontrivial (double-sheeted) manifold. We vindicate the "minimal re-coupling" interaction formula introduced here. We then summarize the pertinent results obtained in [53] for the equivalent Dirac problem of a test point positron in the field of an infinitely massive zGKN singularity: essential selfadjointness of the Hamiltonian; symmetry of its spectrum about zero; the usual Dirac continuum with a gap; a discrete spectrum inside the gap if the ring radius and the coupling constant are small enough — in principle numerically computable by ODE methods using the Chandrasekhar–Page– Toop separation-of-variables method in concert with the Prüfer transform. Here, we also show (formally) that in the limit of vanishing diameter of the ring singularity the positive part of the discrete spectrum reproduces Sommerfeld's fine structure spectrum (with correct Dirac labeling) for Born–Oppenheimer Hydrogen, while the negative part produces the negative of the same (we also explain why the familiar special-relativistic Dirac spectrum only contains half of it). We argue why this result implies that the choice of ring diameter for an electron/anti-electron bi-particle structure should only be a tiny fraction of the reduced Compton wavelength of the electron. We also explain why a study of finite-size effects of the ring singularity on the spectrum using perturbation theory may be problematic, while perturbative computations of effects of a Kerr–Newman-anomalous magnetic moment on the spectrum is presumably feasible. The technical sections 3.3.2, 3.4.1, 3.4.2 are nearly verbatim adapted from [53] and reproduced here for the convenience of the reader.

Section 4: We formulate de Broglie–Bohm laws of motion and orientation for the zGKN-type ring singularity.

Section 5: We conclude with suggesting future work. In particular, we include some speculations about the proper two- and many-body theories consisting entirely of (semi-classically) electromagnetically interacting zGKN ring singularities. The two-body problem is clearly of interest as a putative model for *positronium*, while the many-body theory may offer an intriguing novel explanation of why the particle/anti-particle symmetry is broken in our (part of the) universe.

Appendix: Two appendices supply technical material, one appendix contrasts our work with earlier proposals to link the Kerr–Newman spacetime with "Dirac's equation for the electron."

Almost everywhere in this paper we work in spacetime units in which the speed of light in vacuo c = 1, and in the more mathematical parts we will also set \hbar , Planck's constant divided by 2π , equal to unity; in some physically important formulas we will re-instate both c and \hbar .

2 zGKN spacetimes & electromagnetic fields, and generalizations

2.1 Electromagnetic spacetimes

An electromagnetic spacetime is a triple $(\mathcal{M}, \mathbf{g}, \mathbf{F})$, consisting of a four-dimensional manifold \mathcal{M} , a Lorentzian metric \mathbf{g} on \mathcal{M} (here with signature (+, -, -, -) in line with recent mathematical works on the general-relativistic Dirac equation), and a two-form \mathbf{F} representing an electromagnetic field defined on \mathcal{M} , altogether solving the Einstein–Maxwell equations (in units in which c = 1)

$$R_{\mu\nu}[\mathbf{g}] - \frac{1}{2}R[\mathbf{g}]g_{\mu\nu} = 8\pi G T_{\mu\nu}[\mathbf{g}, \mathbf{F}]$$
(1)

$$\nabla^{\mu}F_{\mu\nu} = 0. \tag{2}$$

Here, $R_{\mu\nu}$ denotes the components of the Ricci curvature tensor and R the scalar curvature of the metric **g**. Finally, $T_{\mu\nu}$ are the components of the trace-free electromagnetic energy(-density)-momentum(-density)-stress tensor:

$$T_{\mu\nu} = \frac{1}{4\pi} \left(F^{\lambda}_{\mu} \star F_{\nu\lambda} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right), \qquad (3)$$

where \star denotes the Hodge duality map. Since **T** is trace-free, viz. $T^{\mu}_{\mu} = 0$, the Ricci scalar R = 0, so that (1) simplifies to

$$R_{\mu\nu}[\mathbf{g}] = 8\pi G T_{\mu\nu}[\mathbf{g}, \mathbf{F}] \tag{4}$$

2.2 A brief recap of the electromagnetic Kerr–Newman spacetimes

The "outer" Kerr-Newman (KN for short) family of spacetimes is a three-parameter family of stationary axisymmetric solutions ($\mathcal{M}, \mathbf{g}, \mathbf{F}$) to the Einstein-Maxwell equations. The three parameters mentioned are the ADM mass (total energy) M, ADM angular momentum per unit mass a, and total charge Q; here, ADM stands for Arnowitt, Deser, and Misner, who defined these quantities in terms of surface integrals at spatial infinity; the charge is similarly defined using Gauss' theorem. Of course, the solution ($\mathcal{M}, \mathbf{g}, \mathbf{F}$) also depends on G, though only in combination with M and Q². The KN metric \mathbf{g} is singular on the timelike cylindrical surface whose cross-section at any fixed t is a circle of (Euclidean) radius $\sqrt{a^2 + \kappa Q^2}$, where $\kappa = 2G$, and G is Newton's constant of universal gravitation. This circle is commonly referred to as the "ring" singularity. The KN electromagnetic field \mathbf{F} is also singular on the same ring as the metric, while asymptotically (near spatial infinity) it becomes indistinguishable from an electric monopole field of moment Q and a magnetic dipole field of moment Qa in Minkowski spacetime.

The causal structure of the maximal-analytical extension of the KN spacetime is quite complex [19] (see also [42, 65]), and depends on the relationship between the parameters: For $a^2 + \kappa Q^2 < \kappa^2 M^2$ (the subextremal case) there is an "ergosphere" region, where due to frame dragging the spacetime is no longer stationary. In addition, there are two horizons. One is an event horizon (boundary of a black hole region) shielding one asymptotically flat end of the manifold from the ring singularity and the acausal region surrounding the latter, and the other a Cauchy horizon, beyond which the initial data do not determine the evolution. For $a^2 + \kappa Q^2 > \kappa^2 M^2$ (the superextremal case) there is still an ergosphere region but there is no event horizon, and thus the ring singularity is naked. The region near the ring is particularly pathological in both sub- and super-extremal cases since it includes closed timelike loops. In the superextremal case the presence of these loops turns the entire manifold into a causally vicious set.

2.3 Zero-*G* limit of Kerr–Newman spacetimes and their electromagnetic fields

It is however possible to rid the KN family from all its causal pathologies mentioned above: take the limit $G \rightarrow 0$ of the KN family in a global chart of asymptotically flat coordinates which respect all the symmetries of the spacetime, while fixing the values of the three parameters M,Q, and a. Since M occurs only in the combination GM, a two-parameter family of spacetimes, depending on a and Q, emerges in the limit, and is denoted as the zero-G Kerr-Newman (zGKN) family. In the zGKN family there are no horizons, no ergosphere regions, no closed timelike loops, and no causally vicious regions. Even though the metrics and fields in this family are still singular on a ring (now of radius |a|), the spacetimes do not suffer from any of these maladies. Indeed, the manifold is locally flat, and the singularity of the metric at a point on the ring is relatively mild, namely conical.

A key feature associated with this ring singularity which survives the zero-G limit is the Zipoy topology of the double-sheeted maximal-analytical extension of the KN family of spacetimes. Remarkably, the entire zGKN spacetime can nevertheless be covered by a single chart of coordinates, named after Boyer–Lindquist (BL) and usually denoted by (t, r, θ, φ) , where the timelike coordinate t and the spacelike coordinate φ are Killing parameters (i.e. $\frac{\partial}{\partial t}$ is a timelike and $\frac{\partial}{\partial \varphi}$ a spacelike Killing field) and (r, θ, φ) are oblate spheroidal coordinates in t =const. slices; see Fig. 3. Each constant-t slice consists of two copies of \mathbb{R}^3 (the sheets) that are cross-linked through two 2-discs at r = 0, as depicted in Fig. 3; one sheet corresponds to r > 0 and the other to r < 0.

The KN electromagnetic field (which is in fact independent of G), is naturally defined on this branched space. To physicists situated far from the ring, the charge carried by it appears positive in one sheet, and negative in the other. Thus, in the same vein in which a Coulomb singularity in Minkowski spacetime is traditionally interpreted as representing an electrically charged point particle, the ring singularity of the Kerr–Newman spacetime in its zero-G limit can be thought of as a particle which, when viewed from spacelike infinity, appears as an electrically charged point particle with a magnetic dipole moment, yet there are two different asymptotically flat ends, and in one such end the particle appears as the anti-particle of what is visible in the other sheet.

The above description of the electromagnetic Kerr-Newman singularity in its zero-G limit makes it plain that general relativity supplies a singularity with an electromagnetic structure which suggests the interpretation of electron and positron as being just two "different sides of the same medal," or in analogy to Heisenberg's iso-spin concept, two different "iso-spin" states of one and the same more fundamental particle. Since Heisenberg's iso-spin referred to nucleons (places in the chart of "isotopes," or rather "isobars"), we don't want to use "iso-spin" for the suggested electron-positron dichotomy. Instead, since the binary character of the Kerr-Newman singularity is associated with the non-trivial topology of its spacetime, we will use the term "topo-spin" (short for "topological

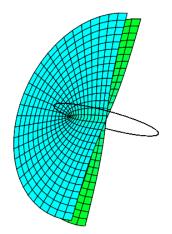


Figure 3: An immersion of a constant- φ section of a constant-t snapshot of the zGKN spacetime together with its ring singularity (parametrized by φ); also shown is an oblate spheroidal coordinate grid — the semi-ellipses stay on a sheet, while the hyperbolas transit from one sheet to the other.

spin"). This term will be vindicated below: we will show that there is a sheet-swap transformation whose derivative induces a representation of the Lorentz group acting on the tangent bundle which is equivalent to the usual spin 1/2 representation.

2.3.1 The metric of the maximal-analytically extended zGKN spacetime

We begin by recalling some standard definitions and results:

Let $\mathcal{M} = (\mathbb{R}^{1,3} \setminus \partial \mathcal{Z}) \sharp (\mathbb{R}^{1,3} \setminus \partial \mathcal{Z})$ be a connected sum of two copies of the Minkowski spacetime from each of which the boundary $\partial \mathcal{Z}$ of a timelike cylinder \mathcal{Z} (2-disc $\times \mathbb{R}$) has been removed, and which are cross-linked through the interior of the cylinders in the manner depicted in Fig. 4 for a constant-time snapshot.

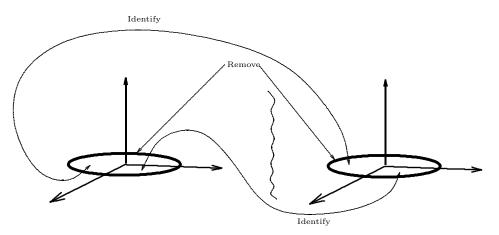


Figure 4: An illustration of the Zipoy topology of the zGKN spacetime at a constant-time snapshot, using two copies of \mathbb{R}^3 . The ring singularity, which appears positively charged in one copy of \mathbb{R}^3 , and negatively charged in the other, is not part of this Riemannian constant-time manifold.

In terms of coordinate charts, the manifold can be described thus. Let a be fixed. Let (t, ϱ, z, φ) denote cylindrical coordinates on $\mathbb{R}^{1,3}$ and let (t, r, θ, φ) be BL coordinates on the zGKN manifold

 \mathcal{M} , with the same (t, φ) as in cylindrical coordinates, and with (r, θ) elliptical coordinates which are related to (ϱ, z) by

$$\varrho = \sqrt{r^2 + a^2} \sin \theta, \qquad z = r \cos \theta.$$

The identification in \mathcal{M} is to be done in such a way that in each fixed timelike half-plane $t = t_0, \varphi = \varphi_0$, one of the sheets is described by $r \ge 0$ and the other by $r \le 0$ (the linking at t =constant is through a double disk at r = 0; cf. Fig. 4) with all the BL coordinates having smooth transitions from one sheet to the other. Note that the coordinate map $(t, r, \theta, \varphi) \mapsto (t, \varrho, z, \varphi)$ is 2 : 1.

We can also view \mathcal{M} as a bundle over the base manifold $\mathbb{R}^{1,3} \setminus \partial \mathcal{Z}$, with the projection map $\Pi : \mathcal{M} \to \mathbb{R}^{1,3} \setminus \partial \mathcal{Z}$ being $\Pi(t, r, \theta, \varphi) = (t, \varrho, z, \varphi)$. The fiber over a point in the base is thus a discrete set of two points; note that Π degenerates at the boundary $\partial \mathcal{Z}$ of \mathcal{Z} ,

$$\partial \mathcal{Z}_t = \{ (t, r, \theta, \varphi) \mid t \in \mathbb{R}, \ r = 0, \ \theta = \pi/2, 0 \le \varphi \le 2\pi \},\$$

where only one point lies above the extended base manifold $\mathbb{R}^{1,3}$ (see Fig. 5). Such a bundle is also known as a "branched cover" of $\mathbb{R}^{1,3}$, with two copies of $\mathbb{R}^{1,3}$ ("branches") which are "branching off each other over a ring."

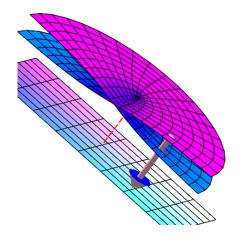


Figure 5: The constant-*t*-constant- φ section of the *zG*KN spacetime \mathcal{M} (with its (r, θ) coordinate grid) as a branched cover over the pertinent constant-*t*-constant- φ section of $\mathbb{R}^{1,3} \setminus \partial \mathcal{Z}$ (with its (ϱ, z) coordinate grid). The endpoints of the dashed line mark the locations of the ring singularity in \mathcal{M} (at $(0, \pi/2)$) and its base point in $\mathbb{R}^{1,3} \setminus \partial \mathcal{Z}$ (at (1,0)); neither are part of the pertinent Lorentz manifolds. The arrow indicates the projection $\Pi : \mathcal{M} \to \mathbb{R}^{1,3} \setminus \partial \mathcal{Z}$.

The pullback of the Minkowski metric η under Π endows \mathcal{M} with a flat Lorentzian metric $\mathbf{g} = \Pi^* \eta$, which solves the Einstein vacuum equations, and which in BL coordinates has the line element

$$ds_{\mathbf{g}}^{2} = dt^{2} - (r^{2} + a^{2})\sin^{2}\theta d\varphi^{2} - \frac{r^{2} + a^{2}\cos^{2}\theta}{r^{2} + a^{2}}(dr^{2} + (r^{2} + a^{2})d\theta^{2}).$$
(5)

Incidentally, the spatial part of the BL coordinate system is one representative of so-called *oblate* spheroidal coordinates. All such coordinate systems differ from each other only in the labeling of the level surfaces; any constant φ section of these consists of confocal ellipses and hyperbolas. Thus, an alternative choice of oblate spheroidal coordinates are the *ring-centered* coordinates, defined by

$$\xi = \frac{r}{a}, \qquad \eta = \cos \theta,$$

and with (t, φ) as before. In these coordinates the line element (5) takes the more symmetric form

$$ds_{\mathbf{g}}^{2} = dt^{2} - a^{2}(1+\xi^{2})(1-\eta^{2})d\varphi^{2} - a^{2}(\xi^{2}+\eta^{2})\left(\frac{d\xi^{2}}{1+\xi^{2}} + \frac{d\eta^{2}}{1-\eta^{2}}\right).$$

The metric **g** is singular on the cylindrical surface $\partial \mathcal{Z}$ whose cross section at any t is the ring $\{r = 0, \theta = \pi/2, \varphi \in [0, 2\pi)\}$. The points on the ring are *conical singularities* for the metric, meaning that the limit, as the radius goes to zero, of the circumference of a small circle that is centered at a point of the ring and is lying in a meridional plane $\varphi = \text{const.}$, is different from 2π — in the case of z*G*KN it is 4π . See [77] for details.

The spacetime $(\mathcal{M}, \mathbf{g})$ is the limiting member, in the limit $G \to 0$, of the Kerr-Newman family of spacetimes. The spacetime $(\mathcal{M}, \mathbf{g})$ introduced above is also the zero-G limit of another family, namely the Kerr family of spacetimes, which outside some ergosphere horizon are stationary, axisymmetric solutions to Einstein's vacuum equations. The Kerr family is simply the limit $\mathbf{Q} \to 0$ of the KN family (in BL coordinates), and as long as only the spacetime structure itself is of interest, we may call $(\mathcal{M}, \mathbf{g})$ the (maximal-analytically extended) zGK spacetime. This zGK spacetime $(\mathcal{M}, \mathbf{g})$ is also the zero-G limit of another family, namely the oblate spheroidal Zipoy family of spacetimes, which are maximal-analytically extended static, axisymmetric solutions to Einstein's vacuum equations and not otherwise related to the KN family except for having the same zero-Glimit (so "zGKN = zGK = zGZ"). In fact, the spacetime $(\mathcal{M}, \mathbf{g})$ introduced above is static, in the sense that it features a timelike Killing field which is everywhere hypersurface-orthogonal. This shows that the stationary Kerr and Kerr-Newman spacetimes become static when $G \to 0$.

2.3.2 The zGKN electromagnetic fields and some of their generalizations

The zGKN spacetime $(\mathcal{M}, \mathbf{g}, \mathbf{F})$ is an electromagnetic spacetime, i.e. it comes already equipped with the electromagnetic field $\mathbf{F}_{\mathrm{KN}} = d\mathbf{A}_{\mathrm{KN}}$ of the Kerr–Newman family,¹² with

$$\mathbf{A}_{\mathrm{KN}} = -\frac{r}{r^2 + a^2 \cos^2 \theta} (\mathbf{Q}dt - \mathbf{Q}a \sin^2 \theta \, d\varphi). \tag{6}$$

The field **F** is singular on the same ring $\{r = 0, \theta = \pi/2, \varphi \in [0, 2\pi)\}$ as the metric, while for r very large it exhibits an electric monopole of strength Q and a magnetic dipole moment of strength Qa; for r very large negative it exhibits an electric monopole of strength -Q and a magnetic dipole moment of strength -Qa. As mentioned in the introduction, this static electromagnetic field was discovered by Appell [1] in 1887, while Sommerfeld [71] realized that it lives on zGKN.

Remark 2.1. The Kerr-Newman spacetime is famously known to have a gyromagnetic ratio Q/M = gQ/2M corresponding to a g-factor of $g_{KN} = 2$, the terminology being borrowed from quantum mechanics and motivated by the facts that the Kerr-Newman spacetime is associated with an ADM spin angular momentum aM, a charge Q, and a magnetic moment Q (as seen asymptotically "from infinity"); see [19, 72, 63]. Since the zGKN spacetime (or rather any of its ∂_t -orthogonal space slices) is static and not "gyrating," it becomes problematic to speak of a gyromagnetic ratio for zGKN; also, since no M features in the zGKN (i.e. zGK) metric, one would need to introduce new notions of "mass and angular momentum of zGKN." One could try to argue that not the spacetime but the ring singularity is gyrating, with a spin angular momentum equal to aM, and with M the inert mass of the ring singularity, yet this has to be taken with a grain of salt, for the electromagnetic field energy of zGKN is infinite, so that according to Einstein's mass-energy equivalence also M ought to be infinite. For a resolution of this mass-energy puzzle in a general

¹²Interestingly, the KN fields are independent of G and therefore survive intact in the zero-G limit.

relativistic spacetime with a single point charge using the nonlinear Einstein–Maxwell–Born–Infeld (and other nonlinear) equations, see [76].

The Kerr–Newman electric field \mathbf{E}_{KN} and the Kerr–Newman magnetic field \mathbf{B}_{KN} are gradients,

$$\mathbf{E}_{\mathrm{KN}} = d\phi_{\mathrm{KN}}, \qquad \mathbf{B}_{\mathrm{KN}} = d\psi_{\mathrm{KN}},$$

where the scalar potentials $\phi_{\rm KN}$ and $\psi_{\rm KN}$ have remarkably simple, and symmetric, expressions in oblate spheroidal (ξ, η, φ) coordinates on \mathcal{N} , the t = 0 slice of zGK, namely¹³

$$\phi_{\rm KN} = \frac{{\rm Q}}{a} \frac{\xi}{\xi^2 + \eta^2}, \qquad \psi_{\rm KN} = \frac{{\rm Q}}{a} \frac{\eta}{\xi^2 + \eta^2}.$$

Note in particular that these two are anti-symmetric with respect to the "toggle" map that swaps the two sheets, viz. $\varsigma : (\xi, \eta) \mapsto (-\xi, -\eta)$. It is therefore evident that in the sheet where $\xi > 0$ the asymptotic behavior of ϕ_{KN} is $\frac{Q}{|\mathbf{q}-\mathbf{q}_{\text{rg}}|}$ while in the other sheet, where $\xi < 0$, the asymptotic behavior becomes $\frac{-Q}{|\mathbf{q}-\mathbf{q}_{\text{rg}}|}$. Thus by Gauss's law the charge carried by the ring is Q in the first sheet and -Q in the second.

Now we note that by the decoupling of spacetime structure from its matter/field content in the zero-G limit, by the linearity of Maxwell's vacuum equations, and by the decoupling of their electric and magnetic subsystems, we can generalize the electromagnetic potential field (7) supported on zGK by adding any almost everywhere (on zGK) harmonic electric or magnetic potential field solving the Maxwell equations on zGK, see [33].

In particular, the KN electromagnetic field can readily be generalized to exhibit a KN-anomalous magnetic moment,

$$\mathbf{A}_{\mathrm{KN}}^{\mathrm{gen}} = -\frac{r}{r^2 + a^2 \cos^2 \theta} (\mathbf{Q}dt - \mathbf{I}\pi a^2 \sin^2 \theta \, d\varphi),\tag{7}$$

with which one can decorate the zGK spacetime of the same ring radius |a|; here, I is the electrical current which produces a magnetic dipole moment $I\pi a^2$ when viewed from spacelike infinity in the r > 0 sheet. Our terminology for the case $I\pi a \neq Q$ is in analogy to the physicists' "anomalous magnetic moment of the electron;" so, the KN-anomalous magnetic moment is $I\pi a^2 - Qa$.

Incidentally, notice that $\mathbf{A}_{\mathrm{KN}}^{\mathrm{gen}}$ in (7) satisfies the Coulomb gauge.

Furthermore, we can add the electric potential of a point charge source. The electrostatic field \mathbf{E}_{pt} generated by a positive point charge of magnitude Q' must be curl-free, and thus is a gradient:

$$\mathbf{E}_{\rm pt} = d\phi_{\rm pt},$$

where $\phi_{\rm pt}$ solves Poisson's equation on \mathcal{N} with a point-source located at $\mathbf{q}_{\rm pt}$,

$$-\Delta_{\mathcal{N}}\phi_{\mathrm{pt}} = 4\pi \mathbf{Q}' \delta_{\mathbf{q}_{\mathrm{pt}}}.$$

Now, \mathcal{N} is a two-sheeted Riemann space branched over the ring, and the fundamental solution of the Laplacian on such a manifold has been known for a long time [43, 62, 25, 30]. It is best described in terms of *peripolar* [61] (sometimes called toroidal) coordinates (ζ, χ, φ). Their definition is as follows: Let **q** be a point in \mathcal{N} , and set $\mathbf{r} = \mathbf{q} - \mathbf{q}_{rg}$ (recall that \mathbf{q}_{rg} is the center of the electron ring). Consider the plane spanned by **r** and \mathbf{n}_{rg} , the normal to the disc spanned by the ring. It intersects the ring at two antipodal points \mathbf{q}_1 and \mathbf{q}_2 , the smaller index always reserved for the

¹³Even though one obtains a remarkable complex structure with these formulas (cf. [77]), the representation of \mathbf{B}_{KN} as a gradient of a scalar potential is problematic at the ring singularity because of the condition that \mathbf{B} be divergence-free.

point closer to \mathbf{q} . Let d_1 and d_2 denote the distances of \mathbf{q} from \mathbf{q}_1 and \mathbf{q}_2 respectively. Then the peripolar coordinate $\zeta := \ln(d_2/d_1) \ge 0$, and the coordinate χ is simply the angle between vectors $\mathbf{q} - \mathbf{q}_2$ and $\mathbf{q} - \mathbf{q}_1$. Note that χ thus defined will be double-valued on ordinary space, since as \mathbf{q} is moved through the ring the value of χ will jump from $-\pi$, its value on the top side of the disc \mathcal{D} , to π , its value on the bottom side of the disc. The angle χ is an example of a multivalued harmonic function in \mathbb{R}^3 , first studied by Sommerfeld [71], who is credited with introducing the concept of a branched Riemann space, i.e. a three-dimensional analog of a Riemann surface, on which multi-valued harmonic functions such as the peripolar χ become single-valued. Other examples of such branched potentials (the term used by Sommerfeld) include the oblate spheroidal coordinate functions ξ and η . Just as in the case of oblate spheroidal coordinates, the system of coordinates $(t, \zeta, \chi, \varphi)$, with φ being the same azimuthal angle introduced before, form a single chart that covers the maximal extension of zGKN, with $-\pi < \chi < \pi$ in one sheet and $\pi < \chi < 3\pi$ in the other sheet of that space. In terms of peripolar coordinates, the electrostatic potential due to the point charge of magnitude Q' is [43, 62, 25, 30]

$$\phi_{\rm pt} = \frac{\mathbf{Q}'}{R} \left(\frac{1}{2} + \frac{1}{\pi} \sin^{-1} \frac{\cos \frac{\chi - \chi_{\rm pt}}{2}}{\cosh \frac{\vartheta}{2}} \right),\tag{8}$$

where

$$R := |\mathbf{q} - \mathbf{q}_{\rm pt}| = \frac{a\sqrt{2}\sqrt{\cosh\vartheta - \cos(\chi - \chi_{\rm pt})}}{\sqrt{\cosh\zeta - \cos\chi}\sqrt{\cosh\zeta_{\rm pt} - \cos\chi_{\rm pt}}},\tag{9}$$

 $\cosh \vartheta := \cosh \zeta \cosh \zeta_{\rm pt} - \sinh \zeta \sinh \zeta_{\rm pt} \cos(\varphi - \varphi_{\rm pt}), \tag{10}$

and where $\mathbf{q}_{\text{pt}} = (\zeta_{\text{pt}}, \chi_{\text{pt}}, \varphi_{\text{pt}})$ is the position of the point charge in peripolar coordinates. Amending this electric potential field to $\mathbf{A}_{\text{KN}}^{\text{gen}}$ yields the *hydrogenic electromagnetic potential* on a zGK spacetime, thus

$$\mathbf{A}_{\rm hyd} = \mathbf{A}_{\rm KN}^{\rm gen} - \phi_{\rm pt} dt; \tag{11}$$

we will use it later to study the Born–Oppenheimer Hydrogen problem with our Dirac equation for a zGKN singularity (with / without anomaly) of charge Q = -e interacting with a positive point charge of magnitude Q' = e supported elsewhere on the zGK spacetime; here, e is the empirical elementary charge used by physicists.

3 Dirac's equation for a zero-*G* Kerr–Newman type singularity

In this section we associate the single-particle Dirac wave function in a consistent manner first with the neutral ring singularity of the topologically non-trivial zGK spacetime consisting of two cross-linked copies of Minkowski spacetime, and subsequently — in a compelling manner — with the charge- and current-carrying ring singularity of the topologically non-trivial zGKN spacetime.

We begin with some group theoretical preliminaries, discussing the spinorial representation of the Lorentz group as well as the sheet swap map associated with the non-trivial topology of the zGK and aGKN spacetimes. This is based on what one does when a Dirac equation is to be formulated for a point particle in the zGK or zGKN spacetimes.

Subsequently we invoke the principle of relativity to show that this formalism also covers the one-body Dirac equation for a "free" ring singularity, i.e. one not interacting with any other electromagnetic object in the manifold (the fact that the zGKN ring singularity carries charge and current does not yet enter the formalism); here we also benefit from the pioneering works of

Schiller [69] and others (see [44], and refs. therein), who first investigated whether non-pointlike, axisymmetric structures can be represented by Dirac bi-spinors.

Then we generalize to the one-body Dirac equation for a zero-G Kerr–Newman singularity which interacts electromagnetically with an additional electromagnetic field that can be supported by the zGK manifold. In particular, this extra field may be generated by a point particle, which will be assumed to have such a large mass that it can be treated in Born-Oppenheimer approximation as infinitely massive and thereby as "classical;" more precisely, its location \mathbf{q}_{pt} (and possibly its magnetic moment $\boldsymbol{\mu}_{\text{pt}}$) enter the equations as parameters, not as operators. This formulation generalizes readily to the situation where an anomalous magnetic moment is added to the KN magnetic moment. Since the ring singularity is not a point, its interaction with the point charge (or any other electromagnetic object, for that matter) is *not* given by a minimal coupling formula that multiplies the ring's charge with the Coulomb potential of the point charge evaluated at the ring's center, but by a "minimal re-coupling" formula; our interaction formula is a natural relativistic extension of the usual formalism employed to calculate the many-charges Coulomb interaction from the classical field-energy integral as carried out, e.g., in [47]. We shall explicitly compute the electromagnetic interaction of the zero-G Kerr–Newman singularity — in fact, its generalization to generate the fields (7) — with a point charge.

We then show that in the limit of vanishing ring radius |a| the Dirac point spectrum reproduces a positive plus a negative Sommerfeld fine structure spectrum (with the correct labeling of the levels) — we also explain, why in the traditional special-relativistic calculations one only obtains half of it.

Finally, we point out problems with perturbation theory as a tool for computing corrections to the Sommerfeld fine structure formula in a "small a" regime, and we also comment on the perturbative approach to compute corrections to the zGKN-Dirac spectrum at finite-a coming from a KN-anomalous magnetic moment.

3.1 Group theoretical considerations

3.1.1 Spinorial Representations of the Lorentz group, and topology of zGKN

Except for the last paragraph, the material in this section is classical and can be found, e.g., in [81], pp.68-77.

Let H(2) denote the Hermitian matrices in $\mathbb{C}^{2\times 2}$. It is a real vector space of dimension four, and a basis is $\{\sigma_{\mu}\}_{\mu=0}^{3}$ where $\sigma_{0} = I_{2}$ and σ_{i} are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
(12)

Let $\mathbb{R}^{1,3}$ denote the Minkowski spacetime, with metric represented by $(\eta) = \text{diag}(1, -1, -1, -1)$. Let the two mappings $\sigma, \sigma' : \mathbb{R}^{1,3} \to H(2)$ be defined by

$$\sigma(X) = X^{\mu}\sigma_{\mu}, \qquad \sigma'(X) = X^{0}\sigma_{0} - \sum_{i=1}^{3} X^{i}\sigma_{i}.$$

Each of these mappings gives rise to a representation of the proper Lorentz group $SO_0(1,3)$ (the connected component of the identity in O(1,3)) by matrices in $SL(2,\mathbb{C})$, in the following way: If $A \in SL(2,\mathbb{C})$, let A^{\dagger} denote its Hermitian adjoint. Now let $Y \in \mathbb{R}^{1,3}$ be such that

$$\sigma(Y) = A\sigma(X)A^{\dagger},$$

Then $Y = L_A X$, where L_A is a member of the proper Lorentz group. Note that this shows $SL(2, \mathbb{C})$ to be the double cover of the proper Lorentz group, because both $A = I_2$ and $A = -I_2$ give $L_A = I_4$.

The maps σ and σ' are chosen in such a way that the two representations they give are inequivalent. This is because there is no matrix $S \in SL(2, \mathbb{C})$ such that $S\sigma(X)S^{-1} = \sigma'(X)$. Note that $\sigma'(X) = \sigma(PX)$ where $P = \begin{pmatrix} 1 & 0 \\ 0 & -I_3 \end{pmatrix}$ is an element of the Lorentz group responsible for *space reflection*. There is no element in $SL(2, \mathbb{C})$ that can represent P, because det P = -1. A similar statement is true about the *time reversal* matrix T = -P.

Let
$$\gamma : \mathbb{R}^{1,3} \to \mathbb{C}^{4\times 4}$$
 be defined as $\gamma(X) = \begin{pmatrix} 0 & \sigma(X) \\ \sigma'(X) & 0 \end{pmatrix}$. For $A \in SL(2,\mathbb{C})$ let $\Lambda_A :=$

 $\begin{pmatrix} A & 0 \\ 0 & (A^{\dagger})^{-1} \end{pmatrix}$. Then one checks that $\Lambda_A \gamma(X) \Lambda_A^{-1} = \gamma(L_A X)$ where L_A is as before. Thus the mapping γ gives another representation of the proper Lorentz group by the special linear group $SL(2,\mathbb{C})$. Let also $\Lambda_P := \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$ and $\Lambda_T := \begin{pmatrix} 0 & -iI_2 \\ iI_2 & 0 \end{pmatrix}$. It is easy to see that $\Lambda_M \gamma(X) \Lambda_M^{-1} = \gamma(MX)$ holds for M = P, M = T, and M = PT. Thus γ gives a representation of the full Lorentz group, in the sense that

$$O(1,3) = \{\Lambda_A, \Lambda_P \Lambda_A, \Lambda_T \Lambda_A, \Lambda_{PT} \Lambda_A \mid A \in SL(2,\mathbb{C})\}.$$

Setting $\gamma(X) = \gamma_{\mu} X^{\mu}$ defines the Dirac matrices $\{\gamma_{\mu}\}_{\mu=0}^{3}$. We have

$$\gamma_0 = \gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \qquad \gamma_i = -\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad i = 1, 2, 3.$$

Also define $\alpha_W^k := \gamma^0 \gamma^k = \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix}$ and $\beta_W := \gamma^0$. Thus $\{\beta_W, \alpha_W^k\}$ provides another basis for the same representation of the Clifford algebra by the γ matrices; they can be re-expressed as

$$\gamma^0 = \beta_W, \gamma^k = \beta_W \alpha_W^k$$

This is called the *spinorial*, or *Weyl*, representation.

A basis for another representation of the Clifford algebra, unitarily equivalent to the above one, is given by the matrices $\beta_{DP} := \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$ and $\alpha_{DP}^k := \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}$. This is called the *standard*, or *Dirac–Pauli*, representation. Note that one still has $\gamma^k = \beta_{DP} \alpha_{DP}^k$; however, γ^0 in the Dirac–Pauli representation is β_{DP} , which is not the same as γ^0 in the Weyl representation, which is the same as β_W .

The above calculation was done on the Minkowski spacetime $\mathbb{R}^{1,3}$. Let \mathcal{M} be any Lorentzian manifold. Then the tangent space and the cotangent space at each point on the manifold are copies of $\mathbb{R}^{1,3}$. Thus all of the above can be replicated on the tangent and cotangent bundles of the manifold.

In particular, let \mathcal{M} be the zGKN or zGK spacetime, and let $\{E_{\mu}\}_{\mu=0}^{3}$ denote an orthonormal basis (with respect to **g**) for $T_{p}\mathcal{M}$. A vector $\Xi \in T_{p}\mathcal{M}$, $\Xi = \Xi^{\mu}\frac{\partial}{\partial x^{\mu}}$ has an expansion in this basis $\Xi = X^{\mu}E_{\mu}$ and identifying (X^{μ}) with a point in the Minkowski spacetime, one has $\sigma(X) = X^{\mu}\sigma_{\mu}$. In this way both the tangent and the cotangent bundle of \mathcal{M} have a representation as order two spinors (i.e. objects with two spinor indices, or in other words, operators that act on order one spinors, to be defined below).

In addition, the sheet swap map $\varsigma : \mathcal{M} \to \mathcal{M}$ acts by $\varsigma(t, \xi, \eta, \varphi) = (t, -\xi, -\eta, \varphi)$; it is a bundle map, i.e. $\Pi \circ \varsigma = \Pi$. It is an isometric involution on \mathcal{M} : its pullback acts on the metric like $\varsigma^* g = g$,

while $\varsigma^2 = id$, and it fixes the ring: $\varsigma|_{\mathcal{R}_t} = id$. Its differential $d\varsigma(p) : T_p\mathcal{M} \to T_{\varsigma(p)}\mathcal{M}$ induces an equivalent representation because:

$$\sigma(d\varsigma(X)) = X^0 \sigma_0 - X^1 \sigma_1 - X^2 \sigma_2 + X^3 \sigma_3 = \sigma_3 \sigma(X) \sigma_3.$$

3.1.2 Bi-Spinors and Sheet Swaps

Again, up to (14) and accompanying text, the material in this section is classical and can be found, e.g., in [18].

In particular, the following basic definitions are due to Cartan [18].

Definition 3.1. A vector w in a vector space V (over \mathbb{C}) on which a non-degenerate bi-linear form \langle , \rangle is defined, is called *isotropic* with respect to that bi-linear form if

$$\langle w, w \rangle = 0$$

(Note that the form is assumed to be bi-linear, not *sesqui*-linear.)

For example, the Minkowski metric η is a non-degenerate bi-linear form on \mathbb{C}^4 . A vector $w \in \mathbb{C}^4$ is thus isotropic with respect to η if $w_0^2 - \sum_{i=1}^3 w_i^2 = 0$, i.e. if it is a (complexified) null vector. Let $w \neq 0$ be such a vector. It is easy to see that $W := \gamma(w)$ will be singular, i.e. det W = 0.

Definition 3.2. A vector $\Psi \in \mathbb{C}^4$ is called a *bi-spinor* if there exists a non-zero vector $w \in \mathbb{C}^4$ isotropic with respect to the Minkowski metric η such that

$$W\Psi = 0.$$

The definition of a bi-spinor makes it clear that it is defined *projectively*, i.e. Ψ is equivalent to $\lambda \Psi$ for $\lambda \in \mathbb{C} \setminus \{0\}$.

Recall that for $M \in O(1,3)$ we have shown that

$$\gamma(Mw) = \Lambda_M \gamma(w) \Lambda_M^{-1},\tag{13}$$

where Λ_M is the matrix corresponding to M in the representation of the Lorentz group given by γ -matrices described above. On the other hand, since M preserves the Minkowski bi-linear form $\eta(Mx, My) = \eta(x, y)$, it thus follows that w is isotropic with respect to η iff Mw is. Moreover it is easy to see that if the bi-spinor Ψ' generates the nullspace of Mw, then we must have

$$\Psi' = \Lambda_M \Psi. \tag{14}$$

This is the rule of transformation of (rank-one) bi-spinors. Comparing (14) with (13) we note that, unlike the spinorial representation $\gamma(w)$ of a vector w, a bi-spinor is transformed by the left action alone, not the conjugate action, of the group. In particular, let M correspond to a space rotation of any angle about any given axis. Then, because of (13), Λ_M would have to correspond to a rotation of half of that angle, and thus by (14) a bi-spinor would be rotated through *half* of that angle. A rotation through the angle 2π , which leaves all vectors $w \in V$ invariant, takes Ψ to $-\Psi$ instead.

Finally, we consider the action of ς on a bi-spinor. As explained above, it corresponds to a sheet swap, which can be alternatively described as replacing each point on one sheet with an associated point on the other sheet reached by looping through the ring once (a 2π circle). Since $\varsigma^2 = id$, two full loopings through the ring brings one back to "square one," which is analogous to a spin-1/2 (bi-)spinor rotation of angle 4π corresponding to a full 2π rotation in Euclidean space. Thus we appropriately may speak of the particle associated to our bi-spinor as having "topo-spin" 1/2. **Remark 3.3.** The "looping through the ring" visualization of topo-spin should not be confused with some kind of rotation in \mathcal{N} , the constant-t snapshot of zGKN; it is independent of the notion of a spinorial representation of the rotation group. In particular, "scalar" particles can have topo-spin, too: a scalar wave function $\Psi \in \mathfrak{L}^2(\mathcal{N}, \mathbb{C})$ can be viewed as depending on $\mathbf{q} \in \mathcal{N}$, equivalently on a vector position $\mathbf{r} \in \mathbb{R}^3$ plus a discrete variable $\varkappa \in \{-1, 1\}$ which indicates on which copy of \mathbb{R}^3 the vector \mathbf{r} lives. The similarity with Pauli's original way of writing spin variables as arguments rather than components of Ψ is evident. Still, the action of the topo-spin operator on such a Ψ (see next subsection) represents a sheet swap, not a rotation.

3.1.3 Generalized Cayley–Klein representation of a bi-spinor

In this subsection we describe a representation of Dirac bi-spinors that is a generalization of the Cayley–Klein representation of Pauli spinors. Therefore we first consider the two-component Pauli spinors.

Let $\psi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^2$ be a (two-component) Pauli spinor field. Set

$$R^2 := \psi^{\dagger} \psi,$$

where for any column vector $\psi \in \mathbb{C}^k$ we have defined $\psi^{\dagger} = \psi^{*t}$ to be the complex conjugatetranspose of ψ . Then the unit spinor $\check{\psi}$ has the following SU(2) (Cayley–Klein) representation [38]:

$$\check{\psi} := \frac{1}{R} \psi = \begin{pmatrix} \cos(\Theta/2)e^{i(\Phi-\Omega)/2} \\ \sin(\Theta/2)e^{i(\Phi+\Omega)/2} \end{pmatrix};$$
(15)

here, for each point in the configuration space, (Φ, Θ, Ω) are a triplet of Eulerian angles¹⁴ corresponding to a rotation $\mathcal{R}(\Phi, \Theta, \Omega) \in SO(3)$; clearly, (Φ, Θ, Ω) are real-valued functions on the one-body configuration space. The above representation is obtained as follows: Given $\psi \in \mathbb{C}^2$, there is a unitary matrix $U^{\psi} \in SU(2)$ such that

$$U^{\psi} \left(\begin{array}{c} 1\\ 0 \end{array}\right) = \check{\psi}.$$

The map

$$(\Phi,\Theta,\Omega) \to U := e^{-i\frac{\Omega}{2}\sigma_3} e^{-i\frac{\Theta}{2}\sigma_2} e^{-i\frac{\Phi}{2}\sigma_3} = \begin{pmatrix} \cos(\Theta/2)e^{i(\Phi-\Omega)/2} & -\sin(\Theta/2)e^{-i(\Phi+\Omega)/2} \\ \sin(\Theta/2)e^{i(\Phi+\Omega)/2} & \cos(\Theta/2)e^{-i(\Phi-\Omega)/2} \end{pmatrix}$$
(16)

is a map from SO(3) into its universal cover SU(2) that takes any such triplet of Euler angles (Φ, Θ, Ω) to (one of the two) SU(2) elements that comprise the inverse image of $\mathcal{R}(\Phi, \Theta, \Omega)$ under the covering map.

Remark 3.4. Incidentally, in [9] this representation of Pauli spinors is used to give an ontological fluid interpretation of Pauli's equation in the spirit of Madelung's fluid interpretation of Schrödinger's equation; subsequently [8] it was noted that it supplies a non-relativistic law for the orientation of a rigid, spinning (spherical) model electron. We also direct the reader to [23] for a related discussion of spinning spherical particles in the context of Nelson's stochastic mechanics.

¹⁴Here and elsewhere in the paper we are using the ZYZ convention for Euler angles, whereby any rotation in \mathbb{R}^3 can be decomposed into a rotation around the *z*-axis, followed by one around the (new) *y*-axis, followed by another one around the (new) *z*-axis

Given any $\psi = \begin{pmatrix} \mathfrak{z}_1 \\ \mathfrak{z}_2 \end{pmatrix} \in \mathbb{C}^2$, one can find a triplet of Eulerian angles (Φ, Θ, Ω) , (unique up to the usual ambiguity in Eulerian angles), such that (15) holds. More precisely, we show below that every non-zero ψ determines an orthonormal frame in \mathbb{R}^3 , denoted by $\{\mathbf{l}(\psi), \mathbf{m}(\psi), \mathbf{n}(\psi)\}$ (cf. [44]) and thus a unique element of the real rotation group SO(3) that takes the standard basis for \mathbb{R}^3 to the basis $\{\mathbf{l}, \mathbf{m}, \mathbf{n}\}$. Set

$$\mathbf{n}(\psi) := \frac{\psi^{\dagger} \boldsymbol{\sigma} \psi}{\psi^{\dagger} \psi} = \frac{1}{|\mathfrak{z}_1|^2 + |\mathfrak{z}_2|^2} \begin{pmatrix} 2 \operatorname{Re}(\mathfrak{z}_1 \mathfrak{z}_2^*) \\ 2 \operatorname{Im}(\mathfrak{z}_1 \mathfrak{z}_2^*) \\ |\mathfrak{z}_1|^2 - |\mathfrak{z}_2|^2 \end{pmatrix};$$

here, the * denotes complex conjugation. It is easy to see that **n** is a unit vector: using the Cayley–Klein form of ψ (15) we obtain

$$\mathbf{n}(\psi) = \left(\begin{array}{c} \sin \Theta \cos \Omega\\ \sin \Theta \sin \Omega\\ \cos \Theta \end{array}\right).$$

Next, let us define the *flip map* $\mathfrak{f}: \mathbb{C}^2 \to \mathbb{C}^2$ as follows:

$$\mathfrak{f}\left(\begin{array}{c}\mathfrak{z}_1\\\mathfrak{z}_2\end{array}\right)=\left(\begin{array}{c}-\mathfrak{z}_2^*\\\mathfrak{z}_1^*\end{array}\right).$$

We note that

$$\mathbf{n}(\mathfrak{f}\psi) = -\mathbf{n}(\psi),$$

which is why \mathfrak{f} is called the flip map. Next we define vectors $\mathbf{l}, \mathbf{m} \in \mathbb{R}^3$ by

$$\mathbf{l} + i\mathbf{m} := rac{(\mathfrak{f}\psi)^{\dagger}\boldsymbol{\sigma}\psi}{\psi^{\dagger}\psi} = \left(egin{array}{c} \mathfrak{z}_1^2 - \mathfrak{z}_2^2 \ i(\mathfrak{z}_1^2 + \mathfrak{z}_2^2) \ -2\mathfrak{z}_1\mathfrak{z}_2 \end{array}
ight),$$

and in terms of the Cayley-Klein parameters:

$$\mathbf{l}(\psi) = \begin{pmatrix} \cos\theta\cos\Omega\cos\Phi + \sin\Omega\sin\Phi\\ \cos\theta\sin\Omega\cos\Phi - \cos\Omega\sin\Phi\\ -\sin\theta\cos\Phi \end{pmatrix}, \qquad \mathbf{m}(\psi) = \begin{pmatrix} \cos\Theta\cos\Omega\sin\Phi - \sin\Omega\cos\Phi\\ \cos\Theta\sin\Omega\sin\Phi + \cos\Omega\cos\Phi\\ -\sin\Theta\sin\Phi \end{pmatrix}.$$

One checks that $\{\mathbf{l}, \mathbf{m}, \mathbf{n}\}$ are indeed orthonormal. Putting the three vectors together gives the rotation matrix $\mathcal{R}(\Phi, \Theta, \Omega)$ mentioned above, which takes the standard frame $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ of \mathbb{R}^3 into the frame $\{\mathbf{l}, \mathbf{m}, \mathbf{n}\}$.

Now let $\Psi \in \mathbb{C}^4$ be a bi-spinor. Thus

$$\Psi = \left(\begin{array}{c} \psi_1\\ \psi_2 \end{array}\right),$$

with $\psi_1, \psi_2 \in \mathbb{C}^2$. Using the Cayley–Klein representations of ψ_1 and ψ_2 , we may write

$$\Psi = \begin{pmatrix} R_1 e^{i\Phi_1/2} \begin{pmatrix} \cos(\Theta_1/2)e^{-i\Omega_1/2} \\ \sin(\Theta_1/2)e^{i\Omega_1/2} \end{pmatrix} \\ R_2 e^{i\Phi_2/2} \begin{pmatrix} \cos(\Theta_2/2)e^{-i\Omega_2/2} \\ \sin(\Theta_2/2)e^{i\Omega_2/2} \end{pmatrix} \end{pmatrix}.$$

Let $R := \sqrt{R_1^2 + R_2^2}$, $\Sigma := 2 \tan^{-1} \frac{R_2}{R_1}$, $S := \frac{1}{2} (\Phi_2 + \Phi_1)$, and $\Phi = \frac{1}{2} (\Phi_2 - \Phi_1)$. Then, $\Psi = Re^{iS} \begin{pmatrix} \cos \frac{\Sigma}{2} e^{-i\Phi/2} \begin{pmatrix} \cos(\Theta_1/2)e^{-i\Omega_1/2} \\ \sin(\Theta_1/2)e^{i\Omega_1/2} \end{pmatrix} \\ \sin \frac{\Sigma}{2} e^{i\Phi/2} \begin{pmatrix} \cos(\Theta_2/2)e^{-i\Omega_2/2} \\ \sin(\Theta_2/2)e^{i\Omega_2/2} \end{pmatrix} \end{pmatrix}.$ (17)

We call this the generalized Cayley-Klein representation of the Dirac bi-spinor Ψ .

We now define the *orientation vector field* of Ψ by

$$\mathbf{n}_{\Psi} := \frac{\Psi^{\dagger} \mathbf{S} \Psi}{\Psi^{\dagger} \Psi} = \cos^2 \frac{\Sigma}{2} \mathbf{n}_1 + \sin^2 \frac{\Sigma}{2} \mathbf{n}_2.$$
(18)

Here, $\mathbf{S} = (S_1, S_2, S_3)^t$, with

$$S_k := \begin{pmatrix} \sigma_k & 0\\ 0 & \sigma_k \end{pmatrix}, \qquad k = 1, 2, 3,$$

while $\mathbf{n}_1 := \mathbf{n}(\psi_1)$ and $\mathbf{n}_2 := \mathbf{n}(\psi_2)$. One readily checks that

$$\|\mathbf{n}_{\Psi}\|^{2} = \frac{1}{2} \left(1 + \cos^{2} \Sigma + (\mathbf{n}_{1} \cdot \mathbf{n}_{2}) \sin^{2} \Sigma\right) \le 1,$$

with equality holding if and only if either $\Sigma = 0$ or π , or if the vectors \mathbf{n}_1 and \mathbf{n}_2 are *parallel*, i.e. $\mathbf{n}_1 \cdot \mathbf{n}_2 = 1$. Of interest is also when this vector field vanishes: $\mathbf{n}_{\Psi} = 0$ iff $\psi_2 = e^{i\Phi} \mathfrak{f} \psi_1$.

Finally, by analogy with the Pauli spinor, for a bi-spinor we can define

$$\mathbf{l}'_{\Psi} + i\mathbf{m}'_{\Psi} := rac{(\mathfrak{F}\Psi)^{\dagger}\mathbf{S}\Psi}{\Psi^{\dagger}\Psi},$$

where

$$\mathfrak{F}\Psi := Re^{iS} \left(\begin{array}{c} \cos\frac{\Sigma}{2}e^{-i\Phi/2} \left(\begin{array}{c} -\sin(\Theta_1/2)e^{i\Omega_1/2} \\ \cos(\Theta_1/2)e^{-i\Omega_1/2} \end{array} \right) \\ \sin\frac{\Sigma}{2}e^{i\Phi/2} \left(\begin{array}{c} -\sin(\Theta_2/2)e^{i\Omega_2/2} \\ \cos(\Theta_2/2)e^{-i\Omega_2/2} \end{array} \right) \end{array} \right).$$

Thus,

$$\mathbf{l}'_{\Psi} = \cos^2 \frac{\Sigma}{2} \, \mathbf{l}_1 + \sin^2 \frac{\Sigma}{2} \, \mathbf{l}_2, \qquad \mathbf{m}'_{\Psi} = \cos^2 \frac{\Sigma}{2} \, \mathbf{m}_1 + \sin^2 \frac{\Sigma}{2} \, \mathbf{m}_2.$$

Unfortunately, the triplet $\{\mathbf{l}'_{\Psi}, \mathbf{m}'_{\Psi}, \mathbf{n}_{\Psi}\}$ forms an orthogonal frame only if $\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{0}$, in general. But, when $\mathbf{n}_1 \times \mathbf{n}_2 \neq \mathbf{0}$, then we can use these vectors to define

$$\mathbf{l}_{\Psi} := \mathbf{n}_1 \times \mathbf{n}_2,$$

which is orthogonal to \mathbf{n}_{Ψ} because \mathbf{n}_{Ψ} is a linear combination of \mathbf{n}_1 and \mathbf{n}_2 , and

$$\mathbf{m}_{\Psi} := \mathbf{n}_{\Psi} \times \mathbf{l}_{\Psi}.$$

The triplet $\{\mathbf{l}_{\Psi}, \mathbf{m}_{\Psi}, \mathbf{n}_{\Psi}\}$ forms an orthogonal frame if $\mathbf{n}_1 \times \mathbf{n}_2 \neq \mathbf{0}$. In each case one can obtain an orthonormal frame by normalization.

After our group-theoretical preparations we are ready to formulate the Dirac equation for a zGKN ring singularity. We begin with the simpler case of a "free" zGKN ring singularity; the interacting case is treated thereafter.

3.2 Dirac's equation for a free z*G*KN ring singularity

Recall that any quantum-mechanical wave equation for a "quantum particle," when formulated "in position space representation," is formulated on the configuration space C of the corresponding "classical (point) particle." For a "free" massive test particle whose classical worldlines (according to Einstein's general theory of relativity) are timelike geodesics in some static background manifold $(\mathcal{M}, \mathbf{g})$ its configuration space is simply a spacelike constant-time slice \mathcal{N} of \mathcal{M} , where "time" parametrizes the timelike Killing field of $(\mathcal{M}, \mathbf{g})$; for a single quantum particle we may more casually write that its wave equation is formulated on $(\mathcal{M}, \mathbf{g})$. In particular, the Dirac equation for a "free" spin-1/2 particle of rest mass m with classical worldline in a spacetime $(\mathcal{M}, \mathbf{g})$, when formulated w.r.t. arbitrary coordinates $(x^{\mu})^3_{\mu=0}$ of \mathcal{M} (with c = 1) reads (cf. [15], [53]):

$$\tilde{\gamma}^{\mu}p_{\mu}\Psi + m\Psi = 0; \tag{19}$$

here, $\Psi = \Psi(x^1, x^2, x^3, x^4)$ is a bi-spinor field on $(\mathcal{M}, \mathbf{g})$, while $p_{\mu} = -i\hbar\nabla_{\mu}$ with ∇ the covariant derivative (on spinors) associated to the spacetime metric \mathbf{g} , and $(\tilde{\gamma}^{\mu})^3_{\mu=0}$ are Dirac matrices associated to this metric, i.e. satisfying the anti-commutation relations

$$\tilde{\gamma}^{\mu}\tilde{\gamma}^{\nu} + \tilde{\gamma}^{\nu}\tilde{\gamma}^{\mu} = 2g^{\mu\nu}\mathbf{1}_{4\times4}.$$
(20)

Now, what is a "free" zGKN ring singularity, and what is its configuration space? These may at first sound like perplexingly difficult questions, but by the principle of relativity the first one has a straightforward answer, and surprisingly also the second question has a simple answer based on the principle of relativity and the group-theoretical considerations of the previous subsection.

First, let's clarify what a "free" zGKN ring singularity is. To see this, start from the picture of a test particle moving freely in \mathcal{N} , its worldline being a timelike geodesic in $(\mathcal{M}, \mathbf{g})$, which for $\mathcal{M} = zGKN$ (and zGK for that matter, too) is a straight line in Euclidean sense, possibly changing sheets by going through the (geometrical) disc spanned by the ring. Any static frame attached to the static zGKN (and zGK) spacetimes is an inertial frame, but any worldpoint of the freely moving test particle is the origin of some other inertial frame, too; so by a simple Lorentz change of inertial frames one can speak of the zGK or zGKN singularity as moving freely w.r.t. an inertial frame in which the point test particle is at the origin.

Next, we clarify what the configuration space is for such a free zGKN ring singularity. This is a more subtle issue, for relative to a Dreibein with origin at $\mathbf{q}_{pt} = (\mathbf{r}_{pt}, \varkappa) \in \mathbb{R}^3 \times \{-1, 1\}$ in a constant-time snapshot \mathcal{N} of zGKN its axisymmetric ring singularity of fixed radius |a| has a geometrical center at $\mathbf{q}_{rg} = (\mathbf{r}_{rg}, \varkappa) \in \mathbb{R}^3 \times \{-1, 1\}$ and a normal $\mathbf{n}_{rg} \in \mathbb{R}^3$ to the disc spanned by the ring; in addition, if one "marks off" a reference point on the ring, then also an azimuthal angle is needed to specify where that point is. Since this requires three spatial plus a two-valued discrete coordinate for \mathbf{q}_{rg} , and two angles for \mathbf{n}_{rg} , and possibly an azimuth, one could come to the conclusion that the configuration space for the ring singularity has to be five- or possibly sixdimensional. However, this counting tacitly assumes a scalar wave function description. Since we are working with the bi-spinorial wave functions of Dirac, the two angles for n_{rg} and the azimuth are encoded in this structure already (see the previous subsection), so that the configuration space for the zGKN ring singularity is indeed merely the set of locations of its center \mathbf{q}_{rg} . Note that also for the standard Dirac point particle one usually speaks of "its classical spin," which is simply formula (18) multiplied by $\hbar/2$, but this way of talking suggests more structure of a structureless point than there really is — for a Dirac point electron, "spin" is purely in the bi-spinorial wave function (acted on by the α_k matrices); by contrast, the same bi-spinorial information acquires ontological meaning for the structured object that the zGKN ring singularity is.

Now we know that the configuration space C for the zGK and zGKN ring singularity is the set of locations \mathbf{q}_{rg} of its center, but which set is this, mathematically? Since the set of all snapshots of where a freely moving test particle can be in \mathcal{N} relative to the ring singularity is \mathcal{N} itself, by relativity (turning the perspective around) the same is true for the center of the ring singularity of zGK and zGKN — so $C = \mathcal{N}$ for a zGKN and zGK ring singularity.

To obtain the domain for the Dirac wave equation of a free zGK or zGKN ring singularity we only have to stack up \mathbb{R} copies of \mathcal{N} (indexed by time) and obtain a manifold which is isomorphic to zGKN (or zGK, without electromagnetism). Because of this isomorphism, the Dirac equation for the bi-spinorial wave function Ψ of a free zGK or zGKN ring singularity "of mass m" is identical to the Dirac equation (19) for a free spin-1/2 particle of mass m on a fixed zGK (or zGKN) background: only the narrative of the variables in the argument of Ψ changes.

The last remark requires some elaboration, though. In our discussion above we have resorted to a notation involving "Euclidean" vectors (plus the discrete variable \varkappa), which are defined relative to some Dreibein attached to an "origin." But clearly, the location of the origin and the orientation of the Dreibein attached to it are merely auxiliary constructs which allow one to invoke vectoralgebra and -calculus. As in Newtonian point mechanics of Newton's universe, the physics does not depend on the choice of origin nor on the Dreibein. In the same vein, the coordinates $q_{pt} =$ $(\mathbf{r}_{pt}, \varkappa) \in \mathbb{R}^3 \times \{-1, 1\}$ enter the Dirac equation (19) only relative to the center and axis of the zGKN ring singularity, and relative to which sheet it is associated with. In fact, by axisymmetry only two coordinates enter the bi-spinor field $\Psi(t, .)$ evaluated at \mathbf{q}_{pt} , namely $|\mathbf{q}_{pt} - \mathbf{q}_{rg}|$ and $(\mathbf{q}_{\rm pt} - \mathbf{q}_{\rm rg}) \cdot \mathbf{n}_{\rm rg}$, which can be expressed in oblate spheroidal coordinates based on the ring, without φ , (see Appendix A). Indeed, since Chandrasekhar [22, 21] showed that the Dirac equation (19) separates in the oblate spheroidal coordinates (BL coordinates) which provide a single chart for its maximal-analytical extension, this has become the coordinate system of choice for essentially all ensuing studies of (19). Clearly, the triplet (r, θ, φ) of oblate spheroidal coordinates of a point in \mathcal{N} are neither its Cartesian coordinates, nor do they refer to a Dreibein attached to the geometrical center of the ring singularity or attached to the point with coordinates (r, θ, φ) . Yet they contain all the relevant information. More to the point, if one knows where a point particle is located in \mathcal{N} relative to the ring singularity, given by the triplet (r, θ, φ) , then one knows where a point of the ring singularity is located relative to the point particle (the φ variable in each case relative to a specifically marked "reference azimuth φ_0 " on the ring singularity); and by axisymmetry, the \mathbb{S}^1 orbit of the φ variable yields the location of the whole ring (both "locations" only modulo a rotation of SO(3) — this is equivalent to giving (r, θ) . And if one knows the location of the ring singularity relative to the point particle, then one can retrieve its center relative to the point particle. Relative to some Dreibein attached to the point particle that center can of course be anywhere on an SO(3) orbit. The SO(3) ambiguity is now fixed by the orientational information extracted from the bi-spinor.

Lastly, we note that instead of merely *re-interpreting* the oblate spheroidal coordinates (r, θ, φ) of a point particle in \mathcal{N} as coordinates of a point on the ring singularity relative to that point particle, as just explained, one can also change variables to other oblate spheroidal coordinates for the center of the ring singularity; see Appendix A.

3.3 Dirac's equation for a zGKN ring singularity (with / without anomaly) interacting with a point charge located elsewhere in the spacetime

To set up the Dirac equation for a spin-half zGKN ring singularity with "inert mass m" and charge Q (as seen asymptotically in the r > 0 sheet) in the electric field of a static point charge Q' located at \mathbf{q}_{pt} in $\mathcal{N} \subset zGKN$, we follow the strategy of the previous subsection and begin by recalling the

Dirac equation for a spin-half test particle of charge Q' and mass m in the zGKN background spacetime of charge Q as seen from infinity in the r > 0 sheet; it was studied in [53], and reads (we allow an anomalous magnetic moment, and set c = 1)

$$\tilde{\gamma}^{\mu} \left(-i\hbar \nabla_{\mu} - \mathbf{Q}' A_{\mathrm{KN}\mu}^{\mathrm{gen}} \right) \Psi + m\Psi = 0, \qquad (21)$$

where the Dirac matrices $(\tilde{\gamma}^{\mu})_{\mu=0}^{3}$ satisfy the same anti-commutation relations as for the "free" Dirac particle, and where $\Psi(t, .)$ and $A_{\mathrm{KN}\mu}^{\mathrm{gen}}(.)$ are evaluated at the location $\mathbf{q}_{\mathrm{pt}} \in \mathcal{N}$ of the point charge, relative to the location of the ring singularity. By following Chandrasekhar's work [22, 21] on the Dirac equation for the "free" Dirac particle (19), Page [66] and Toop [82] showed that (21), with A_{KN} in place of $A_{\mathrm{KN}}^{\mathrm{gen}}$, is separable when the position $\mathbf{q}_{\mathrm{pt}} \in \mathcal{N}$ of the point charge relative to the ring singularity is coordinatized by the oblate spheroidal coordinates (r, θ, φ) .

Now, the conclusions of the previous subsection about the "free" spin-half zGKN singularity should extend to the spin-half zGKN singularity in the field of a point charge: it should be possible to re-interpret (21) as the Dirac equation for a spin-half zGKN ring singularity with inert mass mand charge Q (as seen asymptotically in the r > 0 sheet) and possibly a KN-anomalous magnetic moment, which interacts with the electric field of a static point charge Q' located elsewhere in $\mathcal{N} \subset zGKN$. Indeed, all that would seem to be necessary once again is to re-interpret the oblate spheroidal coordinates (r, θ, φ) of the point charge relative to the ring singularity as coordinates of a point on the ring singularity relative to the point charge. As an important spin-off, the Dirac equation of a spin-half zero-G Kerr-Newman singularity in the field of a point charge would become essentially identical to (21), the Dirac equation of a spin-half point charge in the field of a zero-G Kerr-Newman singularity, which we have studied in [53].

However, all this can only be strictly valid for stationary situations; fortunately, this covers the totality of the bound states associated with the discrete spectrum. Our Dirac equation should still be approximately valid in the quasi-static regime, but not beyond. More on that in section 4.

The reader may object that whereas the minimal coupling interaction term in (21) is easily understandable as describing the effect of $\mathbf{A}_{\mathrm{KN}}^{\mathrm{gen}}$ on a test point charge, it definitely needs to be justified as describing also the effect of the electric field of a static point charge on a "test zGKN singularity" (possibly with KN-anomalous moment) — even for stationary situations.

Remark 3.5. In the interpretation of "a zGKN ring singularity in a given electrostatic field of a point charge" we appropriately should call the term $Q'A_{KN\mu}^{gen}(\mathbf{q}_{pt})\Psi(\mathbf{q}_{pt})$ a "minimal re-coupling term" rather than a "minimal coupling term" — this also pays attention to the fact that in the interpretation of Ψ as bi-spinor wave function of the zGKN ring singularity the coordinates of \mathbf{q}_{pt} relative to the ring singularity are used to locate the ring singularity relative to the point, without requiring a notational change.

In the next subsubsection we shall vindicate the minimal re-coupling term in (21) as the correct interaction term for a spin-half zGKN ring singularity of charge Q (as seen asymptotically in the r > 0 sheet), possibly appended by a Kerr–Newman-anomalous magnetic moment, in the electric field of a static point charge Q' located elsewhere in zGKN.

3.3.1 Vindication of minimal re-coupling

To justify the above form of the Dirac equation for a spin-half zGKN ring singularity with inert mass m and charge Q (as seen asymptotically in the r > 0 sheet) in the electric field of a static point charge Q' located elsewhere in zGKN, we show that the traditional minimal coupling term in (21) which describes the effects of $\mathbf{A}_{\mathrm{KN}}^{\mathrm{gen}}$ on a test point charge is actually obtained from an electromagnetic interaction integral which is completely symmetric in the two objects "point charge" and "zGKN ring singularity" (appended by a KN-anomalous magnetic moment).

We start with the following Dirac equation,

$$\tilde{\gamma}^{\mu} \left(-i\hbar \nabla_{\mu} - \mathcal{P}_{\mu} \right) \Psi + m \Psi = 0, \tag{22}$$

where ∇ and $\tilde{\gamma}$ are as before, and where the four-covector \mathcal{P} is the *interaction energy-momentum "vector"* describing the mutual electromagnetic interaction of the zGKN singularity and the point charge. The interaction energy-momentum vector is defined as follows.

Recall the energy(-density)-momentum(-density)-stress tensor \mathbf{T} with components $T_{\mu\nu}$ given by (3); for our stationary fields it is time-independent, depending only on the space variable \mathbf{s} , and in addition also on $|\mathbf{q}_{\rm pt} - \mathbf{q}_{\rm rg}|, (\mathbf{q}_{\rm pt} - \mathbf{q}_{\rm rg}) \cdot \mathbf{n}_{\rm rg}, a$, and on $\mathbf{Q}\mathbf{Q}'$ and $\mathbf{I}\mathbf{Q}'$ (if a Kerr–Newman-anomalous magnetic moment is added), as parameters. Then \mathcal{P} is by definition the four-vector field integral

$$\mathcal{P}_{\mu} := \text{f.p.} \int_{\mathcal{N}} T_{\mu 0}(\mathbf{q}) d^3 s,$$

where f.p. stands for *finite part*. More to the point, we resort to the early "pedestrian" renormalization recipe, as explained e.g. in [47] for the electrostatic N-body Coulomb interaction, and as explained in our context next. Explicitly, we consider the classical electromagnetic problem of computing, in the quasi-static approximation, the interaction energy of a (generalized) zGKN singularity of charge Q (as seen in the r > 0 sheet) with a point charge Q' that sits elsewhere in the branched Riemann space whose branch curve is the ring singularity. Recall that with Maxwell's vacuum law $\mathbf{D} = \mathbf{E}, \mathbf{H} = \mathbf{B}$, the energy-momentum-stress tensor $T_{\mu\nu}$ of an electromagnetic field tensor $\mathbf{F} = d\mathbf{A}$ is defined to be

$$T_{\mu\nu} = \frac{1}{4\pi} \left\{ F^{\lambda}_{\mu} \star F_{\lambda\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\},\,$$

and thus in particular

$$T_{00} = \frac{1}{8\pi} \left(|\mathbf{E}|^2 + |\mathbf{B}|^2 \right), \qquad T_{0j} = \frac{1}{4\pi} (\mathbf{E} \times \mathbf{B})_j$$

Here for convenience we have already switched to the Euclidean vector notation (the local tangent space of zGK is always Minkowski spacetime); we note that we commit a slight abuse of notation, as we already defined **E** and **B** and **A** as one-forms.

Be that as it may, let the spacetime \mathcal{M} be a copy of zGKN, with its ring singularity centered at \mathbf{q}_{rg} and of radius |a| and orientation \mathbf{n}_{rg} , and let us suppose that a point charge is located at $\mathbf{q}_{pt} \in \mathcal{N}$. We note that \mathbf{E} and \mathbf{B} are of course the *total* fields, computed from \mathbf{A}_{hyd} , so that for the case at hand, namely the (generalized) zGKN singularity plus point charge system, they include self-field contributions from both the (generalized) zGKN singularity and the point charge, in addition to interaction terms. By the linearity of the Maxwell vacuum system, we can write $\mathbf{E} = \mathbf{E}_{pt} + \mathbf{E}_{KN}$ and likewise $\mathbf{B} = \mathbf{B}_{KN}^{\text{gen}}$, where \mathbf{E}_{KN} and $\mathbf{B}_{KN}^{\text{gen}}$ are the (generalized) Kerr–Newman fields associated with (generalized) zGKN, and similarly \mathbf{E}_{pt} is the electric field generated by the static point charge located elsewhere in the zGK spacetime; we have simplified matters by taking $\mathbf{B}_{pt} = 0$, as appropriate for a point charge, and since the addition of a magnetic point dipole field \mathbf{B}_{pt} would in fact be catastrophic. Thus we have

$$4\pi T_{00} = \frac{1}{2} |\mathbf{E}_{\rm pt}|^2 + \frac{1}{2} |\mathbf{E}_{\rm KN}|^2 + \frac{1}{2} |\mathbf{B}_{\rm KN}^{\rm gen}|^2 + \mathbf{E}_{\rm pt} \cdot \mathbf{E}_{\rm KN}.$$

The first three terms in the above are self-energy terms, the integrals of which over the whole space diverges. At the same time, these quantities would be finite for instance if the charges were smeared out over a small region, and then the result would be a (large) constant, but in particular independent of the locations of the particles. Thus, in calculations where only energy differences are important (recall that only differences of eigenvalues of the Hamiltonian have physical spectral significance), these infinite self-energy terms may be ignored, so that only the last term $\mathbf{E}_{\rm pt} \cdot \mathbf{E}_{\rm KN}$ needs to be evaluated. Similarly,

$$4\pi T_{0j} = (\mathbf{E}_{\rm pt} \times \mathbf{B}_{\rm KN}^{\rm gen})_j + (\mathbf{E}_{\rm KN} \times \mathbf{B}_{\rm KN}^{\rm gen})_j,$$

and this time only the first term needs to be computed, the second one integrating to an infinite self-interaction term. Thus,

$$4\pi \mathcal{P}_0 = \int_{\mathcal{N}} \mathbf{E}_{\rm pt} \cdot \mathbf{E}_{\rm KN} d^3 s, \qquad 4\pi \mathcal{P}_j = \int_{\mathcal{N}} (\mathbf{E}_{\rm pt} \times \mathbf{B}_{\rm KN}^{\rm gen})_j d^3 s.$$

Remark 3.6. In our narrative we have been talking about (generalized) zGKN and point charge fields, but the interaction energy-momentum integrals can be used to compute the quasi-static interaction of any two point- or not-point-like electromagnetic objects (subscripts 1 and 2) — in general this will then involve integrands of the type $\mathbf{E}_1 \cdot \mathbf{E}_2$, $\mathbf{B}_1 \cdot \mathbf{B}_2$, $\mathbf{E}_1 \times \mathbf{B}_2$, and $\mathbf{E}_2 \times \mathbf{B}_1$.

Remark 3.7. With this \mathcal{P} in our Dirac equation (22) its gauge invariance seems lost. This seeming contradiction is resolved by noting that energy-momentum conservation still holds if to $T_{0,\mu}$ we add any divergence-free four gradient $\partial_{\mu}\Upsilon$, corresponding to allowing gauge transformations satisfying the Lorenz-Lorentz gauge condition — which one may want to use to keep the theory Lorentz invariant.

We represent

$$\mathbf{E}_{\mathrm{KN}} = -\nabla \phi_{\mathrm{KN}}, \qquad \mathbf{B}_{\mathrm{KN}}^{\mathrm{gen}} = \nabla \times \mathbf{A}_{\mathrm{KN}}^{\mathrm{gen}}.$$

On the other hand, $\mathbf{E}_{pt} = -\nabla \phi_{pt}$, the electrostatic field generated by the point charge located at \mathbf{q}_{pt} , is a gradient, too, with ϕ_{pt} as in (8).

We are now ready to compute the interactions \mathcal{P}_{μ} . We shall show that $\mathcal{P}_{\mu} = Q' A_{KN\mu}^{gen}$.

Proposition 3.8.

$$\mathcal{P}_0 = \mathbf{Q}' \phi_{\mathrm{KN}}(\mathbf{q}_{\mathrm{pt}}).$$

Proof. We need to compute

$$\int_{\mathcal{N}} \nabla \phi_{\rm pt} \cdot \nabla \phi_{\rm KN} d^3 s.$$

The potential ϕ_{pt} is singular at its pole \mathbf{q}_{pt} , while the potential ϕ_{KN} is singular on the ring ($\xi = 0, \eta = 0, 0 \leq \varphi \leq 2\pi$) (in oblate spheroidal coordinates (ξ, η, φ)). The singularities are sufficiently mild so that the their gradients are locally integrable over \mathcal{N} ; i.e., the above energy integral exists. To evaluate it, we excise ϵ -neighborhoods of the singularities and write the above energy integral as limit $\varepsilon \to 0$ of the integral over the remaining domain. We can perform integration by parts and use the fact that ϕ_{pt} and ϕ_{KN} satisfy the Poisson equation with sources supported in the excised regions to convert the integral over the remaining region into a sum of integrals over the excised regions. Using Poisson's equation the evaluation is immediate.

Thus let B_{ϵ} be the Euclidean ball of radius ϵ centered at \mathbf{q}_{pt} , and let T_{ϵ} be the connected sum of two Euclidean tori centered on the ring singularity which are cut and re-glued exactly like \mathcal{N} . Let $\mathcal{N}_{\epsilon} := \mathcal{N} \setminus (B_{\epsilon} \cup T_{\epsilon})$. Then, since ϕ_{KN} is harmonic away from the ring and ϕ_{pt} is finite away from the point charge, and using also that ϕ_{pt} satisfies Poisson's equation with a point source inside B_{ϵ} while ϕ_{KN} is harmonic inside B_{ϵ} , and noting the convention that **n** is always the *outward* normal to the indicated oriented domain of integration, we find

$$\begin{split} \int_{\mathcal{N}_{\epsilon}} \nabla \phi_{\mathrm{pt}} \cdot \nabla \phi_{\mathrm{KN}} d^{3}s &= -\int_{\mathcal{N}_{\epsilon}} \phi_{\mathrm{KN}} \Delta_{\mathcal{N}} \phi_{\mathrm{pt}} d^{3}s + \int_{\partial \mathcal{N}_{\epsilon}} \phi_{\mathrm{KN}} \nabla \phi_{\mathrm{pt}} \cdot \mathbf{n} dS, \\ &= 0 - \int_{\partial B_{\epsilon}} \phi_{\mathrm{KN}} \nabla \phi_{\mathrm{pt}} \cdot \mathbf{n} dS - \int_{\partial T_{\epsilon}} \phi_{\mathrm{KN}} \nabla \phi_{\mathrm{pt}} \cdot \mathbf{n} dS, \\ &= -\int_{B_{\epsilon}} \nabla \phi_{\mathrm{KN}} \cdot \nabla \phi_{\mathrm{pt}} d^{3}s - \int_{B_{\epsilon}} \phi_{\mathrm{KN}} \Delta_{\mathcal{N}} \phi_{\mathrm{pt}} d^{3}s \\ &- \int_{T_{\epsilon}} \nabla \phi_{\mathrm{KN}} \cdot \nabla \phi_{\mathrm{pt}} d^{3}s - \int_{T_{\epsilon}} \phi_{\mathrm{KN}} \Delta_{\mathcal{N}} \phi_{\mathrm{pt}} d^{3}s \\ &= O(\epsilon) + O(\epsilon^{1/2}) - \int_{B_{\epsilon}} \phi_{\mathrm{KN}} \Delta_{\mathcal{N}} \phi_{\mathrm{pt}} d^{3}s \\ &= O(\epsilon^{1/2}) + 4\pi Q' \phi_{\mathrm{KN}}(\mathbf{q}_{\mathrm{pt}}). \end{split}$$

Finally, letting $\varepsilon \to 0$ and dividing by 4π we obtain

$$\mathcal{P}_0 = \mathbf{Q}' \phi_{\mathrm{KN}}(\mathbf{q}_{\mathrm{pt}}).$$

Proposition 3.9. *For* $j \in \{1, 2, 3\}$ *, we have*

$$\mathcal{P}_j = \mathbf{Q}' \mathbf{A}_{\mathrm{KN}}^{gen}(\mathbf{q}_{\mathrm{pt}})_j.$$

Proof. We proceed by analogy to our proof of the previous proposition. Thus, with $\mathbf{E}_{pt} = -\nabla \phi_{pt}$ and $\mathbf{B}_{KN}^{gen} = \nabla \times \mathbf{A}_{KN}^{gen}$, and with $\nabla \cdot \mathbf{A}_{KN}^{gen} = 0$ (the Coulomb gauge), we compute

$$\begin{split} \int_{\mathcal{N}_{\epsilon}} \nabla \phi_{\mathrm{pt}} \times \nabla \times \mathbf{A}_{\mathrm{KN}}^{\mathrm{gen}} d^{3}s &= \int_{\mathcal{N}_{\epsilon}} \left[\nabla \times \left(\phi_{\mathrm{pt}} \nabla \times \mathbf{A}_{\mathrm{KN}}^{\mathrm{gen}} \right) - \phi_{\mathrm{pt}} \nabla \times \nabla \times \mathbf{A}_{\mathrm{KN}}^{\mathrm{gen}} \right] d^{3}s \\ &= \int_{\partial \mathcal{N}_{\epsilon}} \phi_{\mathrm{pt}} \mathbf{n} \times \nabla \times \mathbf{A}_{\mathrm{KN}}^{\mathrm{gen}} dS + \int_{\mathcal{N}_{\epsilon}} \phi_{\mathrm{pt}} \Delta_{\mathcal{N}} \mathbf{A}_{\mathrm{KN}}^{\mathrm{gen}} d^{3}s \\ &= -\int_{\partial \mathcal{B}_{\epsilon}} \phi_{\mathrm{pt}} \mathbf{n} \times \nabla \times \mathbf{A}_{\mathrm{KN}}^{\mathrm{gen}} dS - \int_{\partial \mathcal{T}_{\epsilon}} \phi_{\mathrm{pt}} \mathbf{n} \times \nabla \times \mathbf{A}_{\mathrm{KN}}^{\mathrm{gen}} dS + 0 \\ &= -\int_{\mathcal{B}_{\epsilon}} \nabla \phi_{\mathrm{pt}} \times \nabla \times \mathbf{A}_{\mathrm{KN}}^{\mathrm{gen}} d^{3}s - \int_{\mathcal{T}_{\epsilon}} \left[\nabla \phi_{\mathrm{pt}} \times \nabla \times \mathbf{A}_{\mathrm{KN}}^{\mathrm{gen}} - \phi_{\mathrm{pt}} \Delta_{\mathcal{N}} \mathbf{A}_{\mathrm{KN}}^{\mathrm{gen}} \right] d^{3}s \\ &= O(\epsilon) + O(\epsilon^{1/2}) + \int_{\mathcal{T}_{\epsilon}} \phi_{\mathrm{pt}} \Delta_{\mathcal{N}} \mathbf{A}_{\mathrm{KN}}^{\mathrm{gen}} d^{3}s. \end{split}$$

Once again, the $O(\epsilon^p)$ vanish in the limit $\epsilon \to 0$, thus leaving only the last integral on the torus to consider. But, by the Maxwell–Ampére law,

$$-\Delta_{\mathcal{N}} \mathbf{A}_{\mathrm{KN}}^{\mathrm{gen}} = 4\pi \mathbf{J}_{\mathrm{KN}}^{\mathrm{gen}},$$

where $\mathbf{J}_{\mathrm{KN}}^{\mathrm{gen}}$ is the electric current density (a measure) concentrated on the ring singularity (for this interpretation we need to consider the continuous extension of \mathcal{N} into its ring singularity, which of

course is no longer a differentiable manifold, but a geometrical space. Now we recognize that ϕ_{pt} is nothing but $Q' \times$ the Green function for the negative Laplacian on \mathcal{N} , and so we have

$$\int_{\mathcal{N}} \mathbf{E}_{\rm pt} \times \mathbf{B}_{\rm KN}^{\rm gen} d^3 s = 4\pi \mathbf{Q}' \mathbf{A}_{\rm KN}^{\rm gen}(\mathbf{q}_{\rm pt}).$$
(23)

Dividing by 4π completes the proof.

Thus for \mathcal{P} we have recovered the KN electromagnetic potential (possibly with an anomalous magnetic moment), evaluated at the location of the point charge and multiplied by its charge. This is exactly the minimal coupling formula for the electromagnetic potential of the point charge (treated as a test particle) in the electromagnetic field of the (generalized) KN ring singularity. At the same time, the symmetry w.r.t. the two fields of our interaction integral formula makes it plain that we are really computing the *mutual interaction* of a (generalized) zGKN ring singularity with a point charge, and the same formula should also be used in a true two-body problem, not just the two alternate one-body problems in "external fields" that we have kept talking about.

Remark 3.10. In the non-relativistic limit our result reduces to nothing but Newton's "actio equals re-actio" principle, which implies that the potential energy of object one in the force field of object two equals the potential energy of object two in the force field of object one. With hindsight, one could have elegantly argued this way up front, yet a non-relativistic principle in a relativistic context would certainly have been viewed with some suspicion. Our relativistic energy-momentum formula now completely vindicates this heuristic extension of Newton's principle. In this vein, in the interpretation of (21) where Ψ is a bi-spinor wave function for the zGKN ring singularity the interaction term is properly called a "minimal re-coupling" term, as explained earlier.

3.3.2 The frame formulation of the Dirac equation

Equation (21) is formulated in a nicely compact manner which, however, is not very useful for computations. Using Cartan's frame method (see [15] and refs. therein) one can express the covariant derivative on spinors in terms of standard derivatives:

$$\tilde{\gamma}^{\mu}\nabla_{\mu} = \gamma^{\mu}\mathbf{e}_{\mu} + \frac{1}{4}\Omega_{\mu\nu\lambda}\gamma^{\lambda}\gamma^{\mu}\gamma^{\nu}.$$
(24)

Here $\{\mathbf{e}_{\mu}\}_{\mu=0}^{3}$ is a *Cartan frame*, i.e. an orthonormal frame of vectors spanning the tangent space at each point of the spacetime manifold. We thus have

$$(\mathbf{e}_{\mu})^{\nu}(\mathbf{e}_{\lambda})^{\kappa}g_{\nu\kappa} = \eta_{\mu\lambda},\tag{25}$$

where

$$(\eta) = \text{diag}(1, -1, -1, -1)$$
 (26)

is the matrix of the Minkowski metric in rectangular coordinates. On the one hand, it follows that

$$\tilde{\gamma}^{\mu} = (\mathbf{e}_{\nu})^{\mu} \gamma^{\nu}, \qquad (27)$$

where the γ^{ν} are Dirac matrices for the Minkowski spacetime, satisfying $\gamma^{\nu}\gamma^{\mu} + \gamma^{\mu}\gamma^{\nu} = 2\eta^{\mu\nu}\mathbf{1}_{4\times 4}$. On the other hand, let $\{\boldsymbol{\omega}^{\mu}\}_{\mu=0}^{3}$ denote the *dual* frame to $\{\mathbf{e}_{\mu}\}$, i.e. the orthonormal basis for the cotangent space at each point of the manifold that is dual to the basis for the tangent space:

$$\boldsymbol{\omega}_{\mu}(\mathbf{e}^{\nu}) = \mathbf{e}^{\nu}(\boldsymbol{\omega}_{\mu}) = \delta^{\nu}_{\mu}.$$
(28)

Then the $\Omega_{\mu\nu\lambda}$ are by definition the *Ricci rotation coefficients* of the frame $\{\omega^{\mu}\}_{\mu=0}^{3}$, defined in the following way: Let the one-forms Ω^{μ}_{ν} satisfy

$$d\omega^{\mu} + \Omega^{\mu}_{\nu} \wedge \omega^{\nu} = 0.$$
⁽²⁹⁾

This does not uniquely define the Ω^{μ}_{ν} . However, there exists a unique set of such one-forms satisfying the extra condition

$$\Omega_{\mu\nu} = -\Omega_{\nu\mu},\tag{30}$$

where the first index is lowered by the Minkowski metric: $\Omega_{\mu\nu} := \eta_{\mu\lambda}\Omega^{\lambda}_{\nu}$. Since $\{\omega^{\mu}\}$ forms a basis for the space of one-forms, we have $\Omega_{\mu\nu} = \Omega_{\mu\nu\lambda}\omega^{\lambda}$, which defines the rotation coefficients $\Omega_{\mu\nu\lambda}$.

The Dirac equation (21) on a spacetime $(\mathcal{M}, \mathbf{g})$ with an electromagnetic four-potential \mathbf{A} can thus be written in the following form:

$$\gamma^{\mu} \left(\mathbf{e}_{\mu} + \Gamma_{\mu} - i\mathbf{Q}'\tilde{A}_{\mu} \right) \Psi + im\Psi = 0; \tag{31}$$

here, the Γ_{μ} are connection coefficients,

$$\Gamma_{\mu} := \frac{1}{4} \Omega_{\nu\lambda\mu} \gamma^{\nu} \gamma^{\lambda} = \frac{1}{8} \Omega_{\nu\lambda\mu} [\gamma^{\nu}, \gamma^{\lambda}], \qquad (32)$$

and the \tilde{A}_{μ} are the components of the potential **A** in the ω^{μ} basis, i.e. $\mathbf{A} = \tilde{A}_{\mu}\omega^{\mu}$, or,

$$\tilde{A}_{\mu} := (\mathbf{e}_{\mu})^{\nu} A_{\nu}. \tag{33}$$

Moreover, recall that A_{μ} in (31) is the (μ component of the) generalized zGKN potential evaluated at \mathbf{q}_{pt} while Ψ is the bi-spinor of the zGKN ring singularity evaluated at $(t, \mathbf{q}_{\text{pt}})$, with " \mathbf{q}_{pt} " shorthand for the oblate spheroidal coordinates (r, θ, φ) .

As mentioned earlier, a single chart of oblate spheroidal coordinates (t, r, θ, φ) covers the whole zero-*G* Kerr–Newman spacetime $(\mathcal{M}, \mathbf{g})$, and in these coordinates the electromagnetic Appell– Sommerfeld one-form \mathbf{A}_{KN} is everywhere on $(\mathcal{M}, \mathbf{g})$ given by the simple formula (6); also, the extension to incorporate a Kerr–Newman-anomalous magnetic moment is equally simple, see (7). It is therefore only natural that one would like to write Dirac's equation (21) in these coordinates as well, in the hope of achieving at least some partial separation of variables.¹⁵

However, unlike Cartesian coordinates (x^{μ}) in Minkowski spacetime, oblate spheroidal coordinate derivatives do not give rise to an orthonormal basis for the tangent space at each point of a zero-*G* Kerr spacetime. Thus, to bring (21) into the Cartan form (31) using oblate spheroidal coordinates, one also needs to construct a suitable Cartan frame. Following Chandrasekhar [22, 21], Page [66], Toop [82] (see also Carter-McLenaghan [20]), we introduce a special orthonormal frame $\{\mathbf{e}_{\mu}\}_{\mu=0}^{3}$ on the tangent bundle $T\mathcal{M}$ which is adapted to the oblate spheroidal coordinates in order for the Dirac equation to take a comparatively simple form.

We begin by introducing a Cartan (co-)frame $\{\omega^{\mu}\}_{\mu=0}^{3}$ for the cotangent bundle¹⁶:

$$\boldsymbol{\omega}^{0} := \frac{\boldsymbol{\varpi}}{|\boldsymbol{\rho}|} (dt - a\sin^{2}\theta \, d\varphi), \quad \boldsymbol{\omega}^{1} := |\boldsymbol{\rho}| d\theta, \quad \boldsymbol{\omega}^{2} := \frac{\sin\theta}{|\boldsymbol{\rho}|} (-adt + \boldsymbol{\varpi}^{2}d\varphi), \quad \boldsymbol{\omega}^{3} := \frac{|\boldsymbol{\rho}|}{\boldsymbol{\varpi}} dr, \quad (34)$$

with the abbreviations

$$\varpi := \sqrt{r^2 + a^2}, \quad \rho := r + ia\cos\theta. \tag{35}$$

¹⁵The idea of using special frames adapted to a coordinate system in order to separate spinorial wave equations in those coordinates goes back to Kinnersley [55] and Teukolsky [79].

¹⁶This particular frame is called a *canonical symmetric tetrad* in [20].

Let us denote the oblate spheroidal coordinates (t, r, θ, φ) collectively by (y^{ν}) . Let $g_{\mu\nu}$ denote the coefficients of the spacetime metric (5) in oblate spheroidal coordinates, i.e. $g_{\mu\nu} = \mathbf{g}\left(\frac{\partial}{\partial y^{\mu}}, \frac{\partial}{\partial y^{\nu}}\right)$. One easily checks that written in the $\{\omega^{\mu}\}$ frame, the spacetime line element is

$$ds_{\mathbf{g}}^2 = g_{\mu\nu} dy^{\mu} dy^{\nu} = \eta_{\alpha\beta} \boldsymbol{\omega}^{\alpha} \boldsymbol{\omega}^{\beta}.$$
(36)

This shows that the frame $\{\omega^{\mu}\}_{\mu=0}^{3}$ is indeed orthonormal. With respect to this frame the electromagnetic Sommerfeld potential (7) becomes $\mathbf{A} = \tilde{A}_{\mu} \omega^{\mu}$, with

$$\tilde{A}_{0} = -Q \frac{r}{|\rho|\varpi} - (Q - I\pi a) \frac{a^{2}r\sin^{2}\theta}{\varpi|\rho|^{3}}, \quad \tilde{A}_{1} = 0, \quad \tilde{A}_{2} = -(Q - I\pi a) \frac{ar\sin\theta}{|\rho|^{3}}, \quad \tilde{A}_{3} = 0.$$
(37)

Remark 3.11. We observe that for $Q = I\pi a$, all but one of the quantities A_{μ} vanish, and the non-vanishing one, \tilde{A}_0 , reduces to $-Qr/|\rho|\varpi$.

Next, let the frame of vector fields $\{\mathbf{e}_{\mu}\}$ be the *dual* frame to $\{\boldsymbol{\omega}^{\mu}\}$. Thus $\{\mathbf{e}_{\mu}\}$ yields an orthonormal basis for the tangent space at each point in the manifold:

$$\mathbf{e}_{0} = \frac{\varpi}{|\rho|}\partial_{t} + \frac{a}{\varpi|\rho|}\partial_{\varphi}, \quad \mathbf{e}_{1} = \frac{1}{|\rho|}\partial_{\theta}, \quad \mathbf{e}_{2} = \frac{a\sin\theta}{|\rho|}\partial_{t} + \frac{1}{|\rho|\sin\theta}\partial_{\varphi}, \quad \mathbf{e}_{3} = \frac{\varpi}{|\rho|}\partial_{r}.$$
(38)

Next, the anti-symmetric matrix $(\Omega_{\mu\nu}) = (\eta_{\mu\lambda}\Omega_{\nu}^{\lambda})$ is computed to be

$$(\Omega_{\mu\nu}) = \begin{pmatrix} 0 & -C\omega^0 - D\omega^2 & D\omega^1 - B\omega^3 & -A\omega^0 - B\omega^2 \\ 0 & D\omega^0 + F\omega^2 & -E\omega^1 - C\omega^3 \\ (\text{anti-sym}) & 0 & -B\omega^0 - E\omega^2 \\ 0 & 0 \end{pmatrix},$$
(39)

with

$$A := \frac{a^2 r \sin^2 \theta}{\varpi |\rho|^3}, \ B := \frac{a r \sin \theta}{|\rho|^3}, \ C := \frac{a^2 \sin \theta \cos \theta}{|\rho|^3}, \ D := \frac{a \cos \theta \varpi}{|\rho|^3}, \ E := \frac{r \varpi}{|\rho|^3}, \ F := \frac{\varpi^2 \cos \theta}{|\rho|^3 \sin \theta}.$$
(40)

With respect to this frame on a zero-G Kerr spacetime the covariant derivative part of the Dirac operator (21) can be expressed with the help of the operator

$$\mathfrak{O} := \tilde{\gamma}^{\mu} \nabla_{\mu} = \begin{pmatrix} 0 & \mathfrak{l}' + \mathfrak{m}' \\ \mathfrak{l} + \mathfrak{m} & 0 \end{pmatrix}, \tag{41}$$

where

$$\mathfrak{l} := \frac{1}{|\rho|} \begin{pmatrix} D_+ & L_- \\ L_+ & D_- \end{pmatrix}$$

$$\tag{42}$$

and

$$\mathfrak{l}' := \frac{1}{|\rho|} \begin{pmatrix} D_{-} & -L_{-} \\ -L_{+} & D_{+} \end{pmatrix},\tag{43}$$

with

$$D_{\pm} := \pm \overline{\omega} \partial_r + \left(\overline{\omega} \partial_t + \frac{a}{\overline{\omega}} \partial_{\varphi} \right), \qquad L_{\pm} := \partial_{\theta} \pm i \left(a \sin \theta \, \partial_t + \csc \theta \partial_{\varphi} \right), \tag{44}$$

while

$$\mathfrak{m} := \frac{1}{2} \left[(-2C + F + iB)\sigma_1 + (-A + 2E + iD)\sigma_3 \right]$$

$$= \frac{1}{2|\rho|} \begin{pmatrix} \frac{r}{\varpi} + \frac{\varpi}{\rho^*} & \cot\theta + \frac{ia\sin\theta}{\rho^*} \\ \cot\theta + \frac{ia\sin\theta}{\rho^*} & -\frac{r}{\varpi} - \frac{\varpi}{\rho^*} \end{pmatrix}$$
(45)

and

$$\mathfrak{m}' := \frac{1}{2} \left[(2C - F + iB)\sigma_1 + (A - 2E + iD)\sigma_3 \right] = -\mathfrak{m}^{\dagger}, \tag{46}$$

and the σ_k are Pauli matrices (12), viz.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
(47)

We note that the principal part of $|\rho|\mathfrak{O}$ has an additive separation property:

$$|\rho| \begin{pmatrix} 0 & \mathfrak{l}' \\ \mathfrak{l} & 0 \end{pmatrix} = \left[\gamma^3 \varpi \partial_r + \gamma^0 \left(\varpi \partial_t + \frac{a}{\varpi} \partial_\varphi \right) \right] + \left[\gamma^1 \partial_\theta + \gamma^2 (a \sin \theta \partial_t + \csc \theta \, \partial_\varphi) \right], \tag{48}$$

where the coefficients of the two square-bracketed operators are functions of only r, respectively only θ . Moreover, it is possible to transform away the lower order term in \mathfrak{O} , so that exact separation can be achieved for $|\rho|\mathfrak{O}$. Namely, let

$$\chi(r,\theta) := \frac{1}{2} \log(\varpi \rho^* \sin \theta).$$
(49)

It is easy to see that

$$\mathfrak{m} = \mathfrak{l}\chi, \qquad \mathfrak{m}' = \mathfrak{l}'\chi^*. \tag{50}$$

Let us therefore define the diagonal matrix

$$\mathfrak{D} := \operatorname{diag}(e^{-\chi}, e^{-\chi}, e^{-\chi^*}, e^{-\chi^*})$$
(51)

and a new bi-spinor $\hat{\Psi}$ related to the original Ψ by

$$\Psi = \mathfrak{D}\hat{\Psi}.\tag{52}$$

Denoting the upper and lower components of a bi-spinor Ψ by ψ_1 and ψ_2 respectively, it then follows that

$$(\mathfrak{l} + \mathfrak{m})\psi_1 = (\mathfrak{l} + \mathfrak{m})(e^{-\chi}\hat{\psi}_1) = e^{-\chi} \left[\mathfrak{l} - \mathfrak{l}\chi + \mathfrak{m}\right]\hat{\psi}_1 = e^{-\chi}\mathfrak{l}\hat{\psi}_1,$$
(53)

and similarly

$$(\mathfrak{l}' + \mathfrak{m}')\psi_2 = e^{-\chi^*}\mathfrak{l}'\hat{\psi}_2.$$
(54)

We now put it all together. We set

$$\mathfrak{R} := \operatorname{diag}(\rho, \rho, \rho^*, \rho^*) \tag{55}$$

and note that $|\rho|\mathfrak{D}^{\dagger}\mathfrak{D} = \mathfrak{R}$ while $\mathfrak{D}^{\dagger}\gamma^{\mu}\mathfrak{D} = \gamma^{\mu}$; here, $\mathfrak{D}^{-\dagger}$ is shorthand for $(\mathfrak{D}^{-1})^{\dagger}$. Thus, setting $\Psi = \mathfrak{D}\hat{\Psi}$ in (21) and left-multiplying the equation by the diagonal matrix $\mathfrak{D}' := |\rho|\mathfrak{D}^{-\dagger}$ we conclude that $\hat{\Psi}$ solves a new Dirac equation

$$\left(|\rho|\gamma^{\mu}(\mathbf{e}_{\mu}-i\mathbf{Q}'\tilde{A}_{\mu})+im\mathfrak{R}\right)\hat{\Psi}=0.$$
(56)

3.4 The Dirac Hamiltonian for a general-relativistic zero-gravity Hydrogen atom (in the Born–Oppenheimer approximation)

To make contact with the physical Hydrogen problem, we henceforth identify the electric charge Q of the zGKN spacetime (in its r > 0 sheet) with the electron's empirical negative elementary charge, Q = -e; note that in the other sheet, the charge is automatically that of the positron, +e. The point charge Q' with which the zGKN singularity is interacting electromagnetically is chosen to be the proton's charge, Q' = +e, which we treat as classical; thus we will speak of the general-relativistic zero-gravity Born–Oppenheimer Hydrogen problem. Lastly, the mass parameter m in Dirac's equation we identify with the "empirical mass of the electron, m_e ."¹⁷

3.4.1 The Dirac Equation in Hamiltonian Form

Let us compute the Hamiltonian form of (56). Let matrices M^{μ} be defined by

$$|\rho|\gamma^{\mu}\mathbf{e}_{\mu} = M^{\mu}\partial_{\mu}.$$
(57)

In particular,

$$M^0 = \varpi \gamma^0 + a \sin \theta \, \gamma^2. \tag{58}$$

We may thus rewrite (56) as

$$M^{0}\partial_{t}\hat{\Psi} = -\left(M^{k}\partial_{k} - ie|\rho|\gamma^{\mu}\widetilde{A_{\mathrm{KN}\mu}^{\mathrm{gen}}} + im\mathfrak{R}\right)\hat{\Psi}.$$
(59)

Finally, restoring \hbar and c, and the argument of $A_{\rm KN}^{\rm gen}$, we define

$$\hat{H} := (M^0)^{-1} \left(M^k (-i\hbar\partial_k) - \frac{1}{c} e |\rho| \gamma^{\mu} \widetilde{A_{\mathrm{KN}\mu}^{\mathrm{gen}}}(\mathbf{q}_{\mathrm{pt}}) + mc \mathfrak{R} \right), \tag{60}$$

and can now rewrite the Dirac equation (56) for $\Psi(t, \mathbf{q}_{pt})$ in Hamiltonian form:

$$i\hbar\partial_t\hat{\Psi} = \hat{H}\hat{\Psi}.\tag{61}$$

3.4.2 A Hilbert space for \hat{H}

The correct positive-definite inner product for the space of bi-spinor fields defined on the zGKN spacetime can be extracted from the action for the original Dirac equation (21), which is obtainable from this equation upon left-multiplying it by the conjugate bi-spinor $\overline{\Psi}$, defined as

$$\overline{\Psi} := \Psi^{\dagger} \gamma^0, \tag{62}$$

and integrating the result over a slab of the spacetime. Thus,

$$\mathcal{S}[\Psi] = \int_{t_1}^{t_2} dt \int_{\Sigma_t} \Psi^{\dagger} \gamma^0 \left[\tilde{\gamma}^{\mu} \nabla_{\mu} \Psi + \dots \right] d\mu_{\Sigma_t}, \tag{63}$$

¹⁷Incidentally, one would therefore also like to identify m_e with the "mass of the zGKN singularity," but since the zGKN metric does not contain a mass parameter this would be just a convenient "way of speaking," not a concept backed up by any calculation. In fact, the only "mass" one could possibly assign it by any calculation is its electromagnetic mass, but as for the Coulomb point charge used in the familiar textbook treatments of Dirac Hydrogen, one finds that the electromagnetic field energy of the zGKN ring singularity is infinite. This is not a problem for computing quantum-mechanical spectra as long as G = 0 because then the self-energies only cause an undetectable overall shift of the spectrum, which does not affect the energy differences, viz. the emission / absorption frequencies of the model. The issue becomes important once G is switched on; see the end of our last section.

where $d\mu_{\Sigma_t}$ is the volume element of $\Sigma_t \equiv \mathcal{N}$, any spacelike t = constant slice of zGKN. Using oblate spheroidal coordinates, $d\mu_{\mathcal{N}} = |\rho|^2 \sin \theta d\theta d\varphi dr$, so the natural inner product for bi-spinors on $\Sigma_t = \mathcal{N}$ reads

$$\langle \Psi, \Phi \rangle = \int_{\mathcal{N}} \Psi^{\dagger} \gamma^{0} \tilde{\gamma}^{0} \Phi d\mu_{\mathcal{N}} = \int_{0}^{2\pi} \int_{0}^{\pi} \int_{-\infty}^{\infty} \Psi^{\dagger} M \Phi |\rho|^{2} \sin \theta d\theta d\varphi dr, \tag{64}$$

with

$$M := \gamma^0 \tilde{\gamma}^0 = \gamma^0 \mathbf{e}^0_\nu \gamma^\nu = \frac{\varpi}{|\rho|} \alpha^0 + \frac{a \sin \theta}{|\rho|} \alpha^2.$$
(65)

Here, α^2 is the second one of the three Dirac alpha matrices in the Weyl (spinor) representation

$$\alpha^{k} = \gamma^{0} \gamma^{k} = \begin{pmatrix} \sigma_{k} & 0\\ 0 & -\sigma_{k} \end{pmatrix}, \qquad k = 1, 2, 3,$$
(66)

and for notational convenience the 4×4 identity matrix has been denoted by

$$\alpha^{0} = \begin{pmatrix} \mathbf{1}_{2\times 2} & 0\\ 0 & \mathbf{1}_{2\times 2} \end{pmatrix}.$$
 (67)

Now, let $\Psi = \mathfrak{D}\hat{\Psi}$ and $\Phi = \mathfrak{D}\hat{\Phi}$, with \mathfrak{D} as in (51). Then we have

$$\langle \Psi, \Phi \rangle = \int_0^{2\pi} \int_0^{\pi} \int_{-\infty}^{\infty} \hat{\Psi}^{\dagger} \hat{M} \hat{\Phi} d\theta d\varphi dr, \tag{68}$$

where

$$\hat{M} := \alpha^0 + \frac{a\sin\theta}{\varpi}\alpha^2.$$
(69)

The eigenvalues of \hat{M} are $\lambda_{\pm} = 1 \pm \frac{a \sin \theta}{\varpi}$, both of which have multiplicity 2 and are positive everywhere on this space with Zipoy topology. (Note that $\lambda_{-} \to 0$ on the ring, which is not part of the space time but at its boundary.) We may thus take the above as the definition of a positive definite inner product given by the matrix \hat{M} for bi-spinors defined on the t = const. section of \mathcal{M} , a rectangular cylinder $\mathcal{Z} := \mathbb{R} \times [0, \pi] \times [0, 2\pi]$ with its natural measure:

$$\langle \hat{\Psi}, \hat{\Phi} \rangle_{\hat{M}} := \int_{\mathcal{Z}} \hat{\Psi}^{\dagger} \hat{M} \hat{\Phi} d\theta d\varphi dr.$$
(70)

The corresponding Hilbert space is denoted by H, thus

$$\mathsf{H} := \left\{ \hat{\Psi} : \mathcal{Z} \to \mathbb{C}^4 \mid \|\hat{\Psi}\|_{\hat{M}}^2 := \langle \hat{\Psi}, \hat{\Psi} \rangle_{\hat{M}} < \infty \right\}.$$
(71)

Note that H is not equivalent to $L^2(\mathcal{Z})$ whose inner product has the identity matrix in place of \hat{M} .

After these preparations we are now ready to list our main results which are proved in [53]. Our results about the symmetry of the spectrum are valid with or without the presence of a KN-anomalous magnetic moment, i.e. for any self-adjoint extension of \hat{H} , whatever Q and I. The essential self-adjointness, and the location of essential and point spectra, are stated only for the proper zGKN singularity (i.e. no KN-anomalous magnetic moment) interacting with a point "proton;" however, we conjecture that these results will continue to hold as long as the KN-anomalous magnetic moment is sufficiently small.

3.4.3 Symmetry of the spectrum of \hat{H}

Let $\hat{S} : \mathbb{H} \to \mathbb{H}$ denote the *sheet swap* map $\hat{S}\hat{\Psi}(x) = \hat{\Psi}(\varsigma(x))$ and \hat{K} the complex conjugation operator $\hat{K}\hat{\Psi}(x) = \hat{\Psi}^*(x)$. Let

$$\hat{C} := \gamma^0 \hat{K} \hat{S}.$$

One readily checks that \hat{C} anti-commutes with the full Hamiltonian, and thus if $\hat{\Psi}$ is an eigen-bispinor of \hat{H} with eigenvalue E, then $\hat{C}\hat{\Psi}$ is an eigen-bi-spinor of \hat{H} with eigenvalue -E.

More generally, with the help of the operator \hat{C} , which anti-commutes with any self-adjoint extension of the formal Dirac operator \hat{H} on H, and an argument of Glazman [37], in [53] we prove:

THEOREM 3.12. Let any self-adjoint extension of the formal Dirac operator \hat{H} on H be denoted by the same letter. Suppose $E \in \operatorname{spec} \hat{H}$. Then $-E \in \operatorname{spec} \hat{H}$.

Remark 3.13. The operator \hat{C} should not be confused with the charge conjugation operator

$$\tilde{C} := i\gamma^2 \hat{K}.$$

One easily checks that if $\hat{\Psi}$ solves $i\hbar\partial_t\hat{\Psi} = (\hat{H}_0 + e\mathcal{A})\hat{\Psi}$, then $\tilde{C}\hat{\Psi}$ solves $i\hbar\partial_t(\tilde{C}\hat{\Psi}) = (\hat{H}_0 - e\mathcal{A})(\tilde{C}\hat{\Psi})$. In particular, if $\hat{\Psi}$ is an eigen-bi-spinor of $\hat{H}_0 + e\mathcal{A}$ with eigenvalue E, then $\tilde{C}\hat{\Psi}$ is an eigen-bi-spinor of $\hat{H}_0 - e\mathcal{A}$ with eigenvalue -E (note that the two Hamiltonians here are different).

3.4.4 Essential self-adjointness of \hat{H}

Let \mathcal{Z}^* denote \mathcal{Z} with the ring singularity $\{r, \theta, \varphi | r = 0, \theta = \pi/2\}$ deleted. By adapting an argument of Winklmeier-Yamada [85], in [53] we prove:

THEOREM 3.14. For $Q = -e = I\pi a$, the operator \hat{H} with domain $C_c^{\infty}(\mathcal{Z}^*)$ is e.s.a. on H.

3.4.5 The continuous spectrum of \hat{H}

With the help of the Chandrasekhar–Page–Toop formalism to separate variables, and an argument of Weidmann [83], in [53] we prove:

THEOREM 3.15. For $Q = -e = I\pi a$, the continuous spectrum of \hat{H} on H is $\mathbb{R} \setminus (-m, m)$.

3.4.6 The point spectrum of \hat{H}

Using the Chandrasekhar–Page–Toop formalism, and the Prüfer transform, in [53] we prove:

THEOREM 3.16. Let $Q = -e = I\pi a$. Then, if $|a|m < \frac{1}{2}$ and $e^2 < \sqrt{2|a|m(1-2|a|m)}$, the point spectrum of \hat{H} on H is nonempty and located in (-m, m); the end points are not included. Moreover, the eigenvalues stand in one-to-one correspondence with pairs of heteroclinic orbits connecting two saddle points in two parameter-dependent flows on truncated cylinders.

Remark 3.17. In [53] we surmise that the winding numbers of the heteroclinic orbits enumerate the energy levels (or possibly certain finite families of levels) of the zGKN-Dirac Hamiltonian, and that their right- vs. left-handedness corresponds to positive, resp. negative energy eigenvalues.

Remark 3.18. Resorting to units with \hbar and c restored, our smallness conditions become $2|a| < \frac{\hbar}{mc}$ and $\frac{e^2}{\hbar c} < \sqrt{\frac{2|a|}{\hbar/mc} \left(1 - \frac{2|a|}{\hbar/mc}\right)}$. The condition on the ring diameter 2|a| demands that it be smaller than the electron's Compton wavelength.¹⁸ Now it would seem natural to identify the zero-G Kerr-Newman magnetic moment -ea with the negative of the Bohr magneton (as done for instance by Carter [19]), which yields $2|a| = \frac{\hbar}{mc}$, and then our condition would be violated. However, we shall see below that this choice, while suggestive — and indeed entertained by us for a while — is too naive! Instead, we will compellingly argue that 2|a| should only be about one-thousands of the electron's Compton wavelength. As to the second condition, note that $\frac{e^2}{\hbar c} = \alpha_{\rm s} \approx \frac{1}{137.036}$, and this condition is satisfied for our proposed choice of |a|. More interesting in this regard is the hydrogenic problem where the point "proton" charge e is replaced by the charge Ze of a point "nucleus," with Z > 1, in which case we get a point spectrum in the gap of the continuum as long as $Z < 137.036 \sqrt{\frac{2|a|}{\hbar/mc}} (1 - \frac{2|a|}{\hbar/mc})$; this indicates that our estimate is presumably not sharp (for our upper bound on Z goes $\downarrow 0$ as $|a| \downarrow 0$, while the familiar Dirac bound Z < 137.036 for the existence of a point in the hydrogenic problem on Minkowski spacetime should be obtained instead).

This completes the summary of our main results from [53].

3.4.7 On the computability of the eigenvalues

If the KN-anomalous magnetic moment is absent, i.e. if we assume that $Q = -e = I\pi a$, then $|\rho|\gamma^{\mu} \tilde{A}_{\mu}$ reduces to $|\rho|\gamma^{0} \tilde{A}_{0} = -(Qr/\varpi)\gamma^{0}$, which is a function of only r. The separation of variables Ansatz of Chandrasekhar [21, 22], Page [66], and Toop [82] now yields a system of ordinary differential equations which facilitates the computability of the point spectrum.

Remark 3.19. For the convenience of the reader, in the Appendix B we also recall how to separate the variables for the zGKN Dirac equation, in the absence of the anomalous terms, applying the method of Chandrasekhar–Page–Toop.

However, unlike the familiar ODE system for the Dirac equation of Born–Oppenheimer Hydrogen in Minkowski spacetime, the (say) zGKN-Dirac ODE system does not have a "triangular" structure which would allow its solution one ODE at a time in what may be called the bottom-up direction. Instead, two of the equations are intrinsically coupled, and their joint solution seems to be feasible only numerically with the aid of a computer, for a judicious choice of parameter values of a (see further below) and a few energy eigenvalues, e.g. the positive and negative ground states and a dozen or so excited states. We hope to report on such a study in the not too distant future.

Next we note that for $Q \neq i\pi a$ the quantity $|\rho|\gamma^{\mu} A_{\mu}$ is a function of both r and θ , and unlike the other terms in the Dirac equation (56) it does *not* separate into a sum of two terms each depending only on one of these variables. It follows that the Dirac equation will not be completely separable when the magnetic moment is different from -ea (viz. Qa). A two-dimensional "vector" PDE problem needs to be solved to obtain the energy and angular-momentum eigenvalues. This would with near certainty be feasible only on a computer, too. In the meantime, we have to be content with a few conclusions that can be drawn based on our theorems and some further analysis.

3.4.8 The spectrum in the limit $a \rightarrow 0$ of the electromagnetic spacetime

Under the plausible (but yet to be proved) hypothesis that the limit $a \to 0$ of the zGKN-Dirac spectrum coincides with the spectrum of the Dirac Hamiltonian evaluated directly on the $a \to 0$ limit of the zGKN spacetime (or rather, its t = 0 spacelike slice), we obtain the following results.

First, since the $a \neq 0$ spectrum is symmetric about zero, it is symmetric also in the limit.

¹⁸For convenience, we have dropped the adjective "reduced," even though "Compton wavelength" usually refers to 2π times the reduced expression, i.e. to h/mc.

Second, the continuous spectrum is always $(-\infty, -m] \cup [m, \infty)$, also in the limit $a \to 0$.

Coming thus to the point spectrum, we note that since the spacetime is topologically nontrivial, with a perfect (anti-)symmetry between its two electromagnetic sheets, also the limiting spacetime when $a \to 0$ will be topologically non-trivial, with a perfect (anti-)symmetry between its two electromagnetic sheets; however, the geometry degenerates in the limit $a \to 0$: as the ring radius |a| shrinks to 0, the ring collapses to a point, and the limit $a \to 0$ of the zGKN spacetime thus becomes two copies of Minkowski spacetime with a straight worldline deleted, but with the continuous extension of the two copies into the removed worldline identified at that worldline (to visualize this, remove the time and one space dimension and think of a familiar double cone in \mathbb{R}^3 , pushed flat). Moreover, inspection of the generalized KN field (7) reveals that in the limit $a \to 0$ with all other parameters kept fixed the electromagnetic field becomes two copies of a pure Coulomb field, corresponding to a negative point charge -e in the r > 0 sheet, and a positive point charge +e in the r < 0 sheet. Lastly, we note that the eigenvalue problem now decouples in the sense that the variations can be carried out restricted to either the r > 0 sheet or the r < 0 sheet. Each of these subproblems leads just to the familiar special-relativistic Born–Oppenheimer "Hydrogen" problem, where the quotes around Hydrogen indicate that in one calculation the electron charge -e is replaced by the positron charge +e, and it is well-known that this produces a negative copy of the familiar Born–Oppenheimer Hydrogen spectrum computed from the Dirac equation by Darwin [24] and Gordon [39] with the electron charge -e; cf. [60]. Thus:

In the limit $a \to 0$ the point spectrum of \hat{H} is the union of a Sommerfeld fine structure spectrum with a negative copy of the same (both with proper quantum-mechanical labeling of the eigenvalues).

Remark 3.20. Incidentally, our discussion implicitly explains why in the usual special-relativistic calculations one only obtains half of the symmetric point spectrum: the symmetry of the point spectrum is broken because one tacitly breaks the symmetry of the underlying spacetime by restricting the variations to be supported on only half of it. Of course, nobody at the time of Dirac and Darwin and their contemporaries should have anticipated that!

3.4.9 On the choices of ring radius |a| and ring current I

Temporarily switching to physical units with \hbar and c restored, we note that the most suggestive a-priori choice for |a| would seem to be "half of the electron's Compton wavelength," $|a| = \hbar/2mc$, which is obtained by equating the KN magnetic moment -ea with the negative Bohr magneton $-\hbar e/2mc$; note this implies a > 0. Thus suggestion has been made as early as in [19], where it was observed that the Kerr–Newman spacetime features a g-factor $g_{\rm KN} = 2$. However, as will be explained next, the corresponding zGKN-Dirac spectrum will most likely deviate appreciably from the empirical Hydrogen spectrum for this choice of a. Thus the identification of the zero-GKerr–Newman ring singularity with a binary electron/anti-electron particle structure would seem to receive a devastating blow. However, as already emphasized above, our reasoning will also show that the choice $a = \hbar/2mc$ for the ring size is "too naive."

Namely, we have argued that as $a \to 0$ the positive point spectrum converges to the familiar Sommerfeld fine structure spectrum (with the correct Dirac labeling), and the negative one to the negative thereof. This is consistent with the traditional narrative that in the conventional Dirac Hydrogen problem a *point-like* structureless electron is assumed, with an electric charge -e but no magnetic moment at all. However, the same zero-a Dirac equation, now with a homogeneous magnetic **A** term added to the Hamiltonian, is also known to produce the correct spectrum of a "Dirac electron in an applied homogeneous magnetic field." Thus, the effects traditionally said to come from the "magnetic moment of the electron" are really supplied by the structure of the Dirac matrices (with their physical coefficients) which act on the bi-spinorial Dirac wave function, and not by an additional magnetic dipole structure of the electron.¹⁹ The same will therefore be true in the $a \rightarrow 0$ limit of our zGKN-Dirac Hamiltonian. This in turn implies that for finite-*a* the zGKN magnetic moment -ea will actually make an *additional* contribution to the "magnetic moment of the electron," hence -ea should *not* be identified with the Bohr magneton itself but at most with the "**anomalous magnetic moment of the electron**," to guarantee that the finite-*a* spectrum will continue to agree reasonably well with the Hydrogen spectrum. The upshot is:

The most plausible a-priori choice for the zGKN ring radius |a| is $a \approx 5.83 \times 10^{-4} \hbar/mc (> 0)$.

So much for the a-priori choice of a.

We now come to the choice of the KN-anomalous magnetic moment, and in concert with it a possibly more refined choice of a as well. If, as we have just argued, the finite size of the ring is associated with the anomalous magnetic moment of the electron, then there would seem to be no need to add any further anomalous magnetic moment in form of the KN-anomalous magnetic moment. However, it is certainly conceivable, even likely, that a better agreement of the spectral data with the empirical spectrum will be obtained if both a and I are used as parameters to compute corrections to the Sommerfeld fine structure spectrum. In that case one would set $I\pi a^2 = -5.83 \times 10^{-4} \hbar e/mc$ to identify the total generalized KN magnetic moment with the "electron's anomalous magnetic moment" and use the remaining one-parameter freedom to optimize the generalized zGKN-Dirac spectrum in regard to the empirical one.

3.4.10 On the perturbative computation of finite-a effects on the Hydrogen spectrum

Our proposed a-priori value for a is a tiny value (compared to the "naive choice"); and so, since the relevant atomic length scale for Hydrogen is the Bohr radius $\alpha_s^{-1}\hbar/mc$, one has a tiny dimensionless parameter $a\alpha_s mc/\hbar \approx 5 \times 10^{-6}$. Hence perturbation theory could be sufficient to compute its effects. Furthermore, all eigenfunctions of the Dirac Hamiltonian for Born–Oppenheimer Hydrogen are known when a = 0, so it is tempting to conclude that perturbative computations of finite-a effects on the Hydrogen spectrum for small |a| are straightforward. However, this will not be a problem of the usual perturbation-theoretical type.

In contrast to the standard situation where to a quantum Hamiltonian an extra potential is added which acts on the same Hilbert space domain on which the unperturbed Hamiltonian is defined, here the configuration space itself changes, and with it the Hilbert space domain of the Hamiltonian. For instance, the symmetry of the zGKN-Dirac spectrum implies that one definitely would have to start from the two anti-symmetric copies of the familiar special-relativistic Dirac-"Hydrogen" problem mentioned above and then "switch on" a. After this doubling of the Hydrogen problem one gains the advantage of being able to work with the same global coordinate chart throughout. Unfortunately, the metric and with it the highest order terms in the Hamiltonian change, which raises the problem of dominating them in terms of the unperturbed Hamiltonian.

Be that as it may, as a physicist one may nevertheless want to proceed on a purely formal level and expand²⁰ the Hamiltonian and the metric in powers of a about a = 0, then use formal first-order perturbation formalism to compute corrections of O(a) to the Sommerfeld fine structure spectrum (with correct quantum labeling) — the first order expressions are finite, as we have checked. However, as long as a rigorous justification is missing one needs to be aware of the possibility that this type of naive calculation may lead to incorrect results. If, on the other hand,

¹⁹Of course, since Pauli one knows this, but it is still difficult to not let the conventional narrative of the "magnetic moment of the electron," which suggests it were a true property of the particle like its charge, get in one's way.

²⁰A non-zero radius of convergence is not to be expected, though.

a numerical evaluation of the zGKN-Dirac ODE system should confirm such formal perturbative calculations for small a, then one could have confidence in such calculations, and furthermore the mathematical physicists would be called upon to work out its rigorous foundations. Such numerical studies are currently under way and we hope to report on the results in the near future.

3.4.11 On the perturbative treatment of a KN-anomalous magnetic moment

In contrast to the technical-conceptual problems of treating the small-*a* regime perturbatively about a = 0, once a complete set of eigenfunctions (or at least a sufficiently large subset thereof) has been computed numerically by integrating the zGKN-Dirac ODE system, the regime of a small KN-anomalous magnetic moment can presumably be treated with conventional perturbation theory. We write "presumably," for the addition of a small KN-anomalous magnetic moment to the zGKN-Dirac Hamiltonian brings in the terms with factor $e(-ea - I\pi a^2)$ in the electromagnetic one-form (37), which are more divergent at the ring than the KN terms, and thus need to be controlled by the momentum part of the Hamiltonian. We have not yet established this rigorously, but have made significant progress; we hope to report some definitive results in a follow-up publication.

4 A de Broglie–Bohm law of evolution for the ring singularity

As a quantum-mechanical wave equation our Dirac equation is formulated on the configuration space of the generic position variables of the zGKN-type ring singularity. To make contact with the empirical world one has to specify what the Dirac bi-spinor wave function is supposed to say about the actual position as found in experiments. The conventional, or (in the words of Max Born) orthodox quantum-mechanical interpretation is to assign $\Psi^{\dagger}\Psi$ the meaning of a probability density for this actually found position; however, in order to make it intelligible why Ψ should have such a significance Born argued that Ψ somehow guides the actual particles in their motion in the right way, yet otherwise he did not bother to formulate a suitable guiding equation; cf. [12, 11]. A suitable guiding equation (in the non-relativistic formalism) was supplied by de Broglie [26] and rediscovered and clarified 25 years later by Bohm [6, 7], who subsequently showed that also a relativistic guiding equation for the worldpoint Q^{μ} of a Dirac point particle can be formulated using Dirac's bi-spinors [8]; see [27] for a modern discussion and new insights. For a general introduction into the de Broglie–Bohm theory, see the books [8, 45, 29], each one of which offers a somewhat different perspective. We also recommend [28] for a collection of profound papers on the de Broglie–Bohm type foundations of quantum mechanics, and see also [51] for a review of common misunderstandings regarding this theory.

In the following we explain that a de Broglie–Bohm type law can be formulated for the actual position of the ring singularity relative to the fixed point charge with a Dreibein attached. To obtain this (sometimes so-called) de Broglie–Bohm–Dirac law of motion for the zGKN-type ring singularity, we resort again first to the alternate (and main-stream) interpretation of our Dirac equation (which we use in [53]) as the bi-spinor wave equation for a spin-1/2 point fermion in the zGKN spacetime which interacts with its ring singularity electromagnetically.

4.1 The guiding laws for a Dirac point (anti-)electron in the zGKN spacetime

4.1.1 The four-current and velocity vector fields of Ψ

Associated to a bi-spinor field Ψ is the four-vector field \mathbf{j}_{Ψ} (orthodoxly called *quantum probability* four-current) with components

$$j^{\mu}_{\Psi} := \overline{\Psi} \gamma^{\mu} \Psi$$

equivalently,

$${f j}_{\Psi}=\left(egin{array}{c} \Psi^{\dagger}\Psi\ \Psi^{\dagger}m{lpha}\Psi\end{array}
ight)$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)^t$, with

$$\alpha_i := \left(\begin{array}{cc} \sigma_i & 0\\ 0 & -\sigma_i \end{array} \right).$$

Evaluating on the generalized Cayley–Klein representation (17) of Ψ , we have

$$\mathbf{j}_{\Psi} = R^2 \left(\begin{array}{c} 1 \\ \cos^2 \frac{\Sigma}{2} \mathbf{n}_1 - \sin^2 \frac{\Sigma}{2} \mathbf{n}_2 \end{array} \right).$$

Now we define the density ρ_{ψ} and velocity field \mathbf{v}_{ψ} associated to Ψ as follows:

$$\rho_{\psi} := R^2, \qquad \mathbf{v}_{\psi} := \cos^2 \frac{\Sigma}{2} \mathbf{n}_1 - \sin^2 \frac{\Sigma}{2} \mathbf{n}_2.$$

We then have $(j^{\mu}) = (\rho_{\psi}, \rho_{\psi} \mathbf{v}_{\psi})^t$. One readily checks that

$$\|\mathbf{v}_{\psi}\|^{2} = \frac{1}{2} \left(1 + \cos^{2} \Sigma - (\mathbf{n}_{1} \cdot \mathbf{n}_{2}) \sin^{2} \Sigma\right) \le 1,$$

with equality if and only if either $\Sigma = 0$ or π ; or if \mathbf{n}_1 and \mathbf{n}_2 are *anti-parallel*, i.e. $\mathbf{n}_1 \cdot \mathbf{n}_2 = -1$. Therefore the current (j^{μ}) is always *causal*, i.e. either timelike or null, since

$$\eta_{\mu\nu} j^{\mu} j^{\nu} = \rho_{\psi}^{2} (1 - \|\mathbf{v}_{\psi}\|^{2}) \ge 0.$$
(72)

Moreover, the case of equality is exceptional, see [78].

Remark 4.1. Note that (72) supplies Einstein's relativistic gamma factor of a massive point particle moving at the speed \mathbf{v} . More precisely,

$$\gamma[\mathbf{v}_{\psi}] = \frac{1}{\sqrt{1 - |\mathbf{v}_{\psi}|^2}} = \frac{\rho_{\psi}}{\sqrt{\eta_{\mu\nu} j^{\mu} j^{\nu}}}.$$
(73)

By a similar analysis $\mathbf{v}_{\psi} = 0$ if $\mathbf{n}_1 = \mathbf{n}_2$ and $\Sigma = \frac{\pi}{2}$, or equivalently if $\psi_2 = e^{i\Phi}\psi_1$.

4.1.2 The de Broglie–Bohm-type law of motion for the point positron

With our present choice of charges Q = -e and Q' = e, and with $m = m_e$ the electron mass, this traditional interpretation of the Dirac equation is that of Dirac's point positron in the field of an infinitely massive zGKN ring singularity. The de Broglie–Bohm–Dirac law of motion for the four components $Q^{\mu}(\tau)$ of the actual worldpoint of the point positron then reads

$$\frac{dQ^{\mu}}{d\tau} = \frac{1}{\sqrt{\eta_{\alpha\beta}j^{\alpha}j^{\beta}}} \overline{\Psi}\gamma^{\mu}\Psi \bigg|_{\{q^{\kappa}=Q^{\kappa}\}},\tag{74}$$

or equivalently,

$$\left(\frac{dQ^{\mu}}{d\tau}\right)_{\mu=0}^{3} = \left.\frac{1}{\sqrt{\eta_{\alpha\beta}j^{\alpha}j^{\beta}}} \left(\begin{array}{c}\Psi^{\dagger}\Psi\\\Psi^{\dagger}\alpha\Psi\end{array}\right)\right|_{\{q^{\kappa}=Q^{\kappa}\}};\tag{75}$$

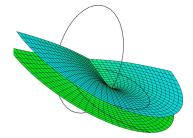


Figure 6: An illustration of a fictitious de Broglie–Bohm world line of a "traditional Dirac point positron," projected onto a constant- φ section of a constant-t snapshot of the zGKN spacetime \mathcal{M} , immersed into three-dimensional Euclidean space; again, the two sheets of this flat Sommerfeld space are slightly bent for the purpose of visualization. Shown is also the ring singularity of the constant-t snapshot of \mathcal{M} .

here, τ is the proper-time variable along the worldline of that point. An illustration of a (fictitious) trajectory of such a de Broglie–Bohm motion of a Dirac point positron in the zGKN spacetime with generic oblate spheroidal coordinates (t, r, θ, φ) is shown in Fig. 6.

If j^{μ} is a null vector then (75) is ill-defined. In that case, for some affine parameter s,

$$\frac{dQ^{\mu}}{ds} = \overline{\Psi}\gamma^{\mu}\Psi\big|_{\{q^{\kappa}=Q^{\kappa}\}}.$$
(76)

Since this case is exceptional [78], it will not be considered further.

4.1.3 The Dreibein attached to the Dirac point positron

As shown in section 3, Ψ evaluated at the worldpoint $(Q^{\mu})(\tau) = (t, r, \theta, \varphi)_{\tau}$ of the point positron (in the interpretation of [53]) defines a normalized Dreibein "attached" to the point positron at $(Q^{\mu})(\tau)$, consisting of the the normalized versions of the orientation vector field \mathbf{n}_{Ψ} given by (18), and its orthogonal complementary vector fields \mathbf{l}_{Ψ} and \mathbf{m}_{Ψ} (constructed in an earlier subsection of section 3), all evaluated at the worldpoint $(Q^{\mu})(\tau)$.

Explicitly, expressing \mathbf{n}_{Ψ} in terms of Ψ (see section 3.1.3), and evaluating at $(Q^{\mu})(\tau)$, we define

$$\mathbf{N}(\tau) = \frac{\Psi^{\dagger} \mathbf{S} \Psi}{\Psi^{\dagger} \Psi} \Big|_{(t,\mathbf{q})=(Q^{\mu})(\tau)} = \left(\cos^2(\frac{1}{2}\Sigma) \mathbf{n}_1 + \sin^2(\frac{1}{2}\Sigma) \mathbf{n}_2 \right) \Big|_{(t,\mathbf{q})=(Q^{\mu})(\tau)}, \tag{77}$$

and we let $\mathbf{\hat{N}}(\tau) = \mathbf{N}(\tau)/\|\mathbf{N}(\tau)\|$ denote the actual unit normal vector; similarly we define $\mathbf{L}(\tau)$ and $\mathbf{M}(\tau)$ as the evaluation of \mathbf{l}_{Ψ} and \mathbf{m}_{Ψ} at the worldpoint $(Q^{\mu})(\tau)$, and set $\mathbf{\check{L}}(\tau) = \mathbf{L}(\tau)/\|\mathbf{L}(\tau)\|$ and $\mathbf{\check{M}}(\tau) = \mathbf{M}(\tau)/\|\mathbf{M}(\tau)\|$. Then at each τ , the Dreibein $(\mathbf{\check{N}}(\tau), \mathbf{\check{L}}(\tau), \mathbf{\check{M}}(\tau))$ defines a unique element of SO(3), say $\mathcal{R}(\tau)$, such that $\mathbf{\check{L}}(\tau) = \mathcal{R}(\tau)\mathbf{\check{L}}(0)$, $\mathbf{\check{M}}(\tau) = \mathcal{R}(\tau)\mathbf{\check{M}}(0)$, and $\mathbf{\check{N}}(\tau) = \mathcal{R}(\tau)\mathbf{\check{N}}(0)$.

4.2 The law of motion for center and orientation of the ring singularity relative to the inertial Dreibein of an infinitely massive point proton at rest

We want to re-interpret the guiding law (75) as supplying also the de Broglie-Bohm-Dirac motion of a spin-1/2 zGKN-type ring singularity of mass $M = m (= m_e)$ and charge Q = -e (as seen in the r > 0 sheet) which interacts with an infinitely massive point "proton" of charge Q' = e. Indeed, if in the interpretation of [53] $(Q^{\mu})_0 = (t, r, \theta, \varphi)_0$ is the initial worldpoint of the point positron, with $\tau = 0$ (say), and $(Q^{\mu})(\tau) = (t, r, \theta, \varphi)_{\tau}$ is its evolved worldpoint at proper time τ , then, as explained in subsection 3.2, each quadruplet of these BL coordinates implicitly also describes the actual location of the zGKN-type ring singularity of mass m and charge -e relative to the position of that point "positron" — which in the interpretation of this paper becomes an infinitely massive point "proton" with a fixed location — with a Dreibein attached to it which is always a translate of the Dreibein at $\tau = 0$.

However, since the Dirac equation for a point positron in the static zGKN spacetime the Dirac bi-spinor also supplies a frame for the point positron, when re-interpreting the Dirac equation in our sense as an equation for a zGKN-type ring singularity moving with respect to a point proton with an inertial α frame attached to it, it is clear that there will be a contribution to the velocity of the center of the ring singularity coming from $\mathcal{R}(\tau)$. More precisely, in the interpretation of our current paper, the ring normal $\check{N}_{rg}(\tau)$ now is evolved by the inverse rotation, viz.

$$\check{\mathbf{N}}_{\rm rg}(\tau) = \mathcal{R}^{-1}(\tau)\check{\mathbf{N}}_{\rm rg}(0).$$
(78)

This is the law of evolution for the orientation of the ring singularity. The position of the center of the moving ring singularity, $\mathbf{q}(\tau)$, w.r.t. the α frame, is simply

$$\mathbf{q}(\tau) = -\mathcal{R}^{-1}(\tau)\mathbf{Q}(\tau). \tag{79}$$

Here, τ is still the proper time along the world line of the "Dirac point positron" from the traditional interpretation; yet since $Q^0(\tau) = t$, we can express the evolutions in terms of the reference time t of the α frame, which coincides with the time t of the BL coordinates.

Remark 4.2. The validity of this re-interpretation of the law of motion is restricted to quasi-static motions relative to the point charge.

4.3 Comments on the de Broglie–Bohm-type laws

4.3.1 Stationary states

Interestingly, in contrast to the non-relativistic de Broglie–Bohm law of electron motion generated from a Schrödinger wave function ψ which yields that for eigenstates ψ the electron sits still (relative to the point "proton"),²¹ the de Broglie–Bohm–Dirac law (75) and the orientation law (77) together imply that even for eigenstate bi-spinor wave functions Ψ the ring singularity generally does not sit still (relative to the point "proton"). A dynamical ring singularity as source of the

²¹Many physicists have expressed their unease upon learning that de Broglie–Bohm theory predicts that particles don't move when guided by an eigenstate Schrödinger wave function. However, when compared with Bohr's famous postulate that the accelerated point charges in the stationary Kepler orbits of his model of the Hydrogen atom move but don't radiate, thereby denying the predictions of classical electromagnetic theory for such motions even though classical electromagnetism was used in his model to compute the interactions, it seems to us to actually be an asset of the de Broglie–Bohm theory that it resolves Bohr's dilemma in such a clean way, i.e. in complete agreement with the Maxwell–Lorentz field equations.

Maxwell equations on a zGK spacetime would generally be expected to emit electromagnetic radiation; note that in our formulation of the general-relativistic zero-G Hydrogen problem in the Born–Oppenheimer approximation radiation is neglected because we are using the quasi-static approximation for the classical electromagnetic fields to compute the interaction energy-momentum vector \mathcal{P} . So one tentative conclusion is that the inclusion of radiation effects, even if only semiclassically, will possibly destroy most of the energy eigenvalues except perhaps the positive and negative "ground states." A similar phenomenon is known from so-called "non-relativisite QED;" in the Hydrogen problem for example, the quantum-mechanical Pauli Hamiltonian for an electron in the Coulomb field of a point proton is additionally coupled to the quantized radiation field, and only the ground state eigenvalue of the Pauli Hamiltonian survives this radiation coupling.

However, at the semi-classical level of electromagnetic coupling that we use here, another possibility is conceivable. Namely, one readily computes that for any solution to Dirac's equation,

$$\mathbf{v}_{\psi} \cdot \mathbf{n}_{\psi} = \cos^4 \frac{1}{2} \Sigma - \sin^4 \frac{1}{2} \Sigma = \cos \Sigma,$$

so that \mathbf{v}_{ψ} is orthogonal to \mathbf{n}_{Ψ} if $\Sigma = \pm \frac{\pi}{2}$, which for non-zero ψ_1 and ψ_2 is equivalent to $|\psi_1| = |\psi_2|$. It is readily checked (see Appendix B) that for eigenstate bi-spinor wave functions Ψ , indeed one has $\Sigma = \pm \frac{\pi}{2}$. Now for any actual quadruplet (t, r, θ, φ) the evaluation of \mathbf{n}_{Ψ} can be rotated by an element of SO(3) to become parallel to \mathbf{n}_{rg} computed from this quadruplet (we used this fact already once before, to fix the initial conditions; see above). By the stationarity of the bispinor wave function, \mathbf{n}_{Ψ} evaluated with this quadruplet (t, r, θ, φ) will then remain parallel to \mathbf{n}_{rg} computed from this quadruplet for all time. Now, Bohm and Hiley [8] showed in the Pauli limit that the de Broglie–Bohm–Dirac velocity is a vector sum of the de Broglie–Bohm expression one obtains with the Schrödinger equation, plus a gyration that is additionally supplied by the bi-spinor structure of the Dirac equation. Since, as mentioned above, the (what may be called) de Broglie-Bohm–Schrödinger term vanishes for an eigenstate, only the gyrational motion remains. We surmise that this will be so also in the fully relativistic expression; we have yet to confirm this. But suppose this surmise is correct, then by the axisymmetry of the problem, with this quadruplet interpreted as the actual position of a point "positron" (using again the usual interpretation) we conclude that the point positron moves in a circle parallel to the ring singularity. In the alternate interpretation put forward in the present paper, the point charge is not a point positron but an infinitely massive un-accelerated point "proton," so from the perspective of that point "proton" the ring singularity now gyrates stationarily about its axis of symmetry, and therefore generates only the static electromagnetic field used to compute the interaction term \mathcal{P} and no classical electromagnetic radiation fields, in complete agreement with our stationarity assumption.

4.3.2 Quasi-Static Motions

In a non-stationary state the re-interpretation of our Dirac equation as covering the motion of a zGKN ring singularity relative to an infinitely massive point proton with Dreibein attached is restricted to the regime of quasi-static motions. This means that the absolute velocities obtained from the gyrational motion need to be much smaller than the speed of light. We now give a rough estimate which shows that for gentle non-stationary perturbations of the Hydrogen ground state we are in the quasi-static regime.

Thus, we recall that the law for the evolution of the orientation vector can be decomposed into a sum of the familiar Larmor formula plus a quantum contribution which contains $|\Psi|^2$. We find that for the KN electromagnetic fields a Bohr distance away from the ring singularity the Larmor frequency times the Bohr distance gives a speed c_{Larmor} of roughly

$$c_{\rm Larmor} \approx 10^{-3} \alpha_{\rm S}^3 c$$

while the quantum contribution is roughly

 $c_{\text{quantum}} \approx \alpha_{\text{S}} c.$

So the speed of the ring singularity relative to a point proton with fixed Dreibein, the α frame, attached is, in such a situation, roughly one-hundredth the speed of light and thus "quasi-static."

4.3.3 A Feynman–Stückelberg dilemma avoided

A final remark concerns a dilemma for the de Broglie–Bohm theory when Dirac's equation is given Dirac's orginal point electron interpretation or the Stückelberg–Feynman interpretation. Namely the fact that with general initial conditions for Ψ both the putative "electronic bi-spinors" and the "positronic bi-spinors" (according to the decomposition of Hilbert space into positive and negative subspaces) enter j^{μ} — and thus also the guiding equation (75) for a single point — is perplexing with these interpretations of Dirac's equation, while it is completely natural in our interpretation of the Dirac particle as having a bi-particle structure in which electron and anti-electron are merely the "two different sides of the same medal," realized through a moving zero-G Kerr–Newman type ring singularity for which we have formulated a Dirac equation.

5 Outlook

The most important task now would seem to be to numerically evaluate the zGKN-Dirac spectrum for the pertinent choice of a and see whether the modifications of the Sommerfeld spectrum are physically reasonable. Since the QED corrections like the Lamb shift are spectacular, it would be unwise to expect too much, but it's certainly an important problem to compute the spectrum quantitatively.

Other important tasks include the discussion of the Hamiltonian in the presence of a zGKNanomalous magnetic moment; as pointed out, we expect that essential self-adjointness holds if the coupling constant $(Qa - I\pi a^2)e$ is small in magnitude. It is also an interesting question whether more than one self-adjoint extension exists for sufficiently large coupling constant, or none at all. Incidentally, note that our essential self-adjointness result for the Dirac Hamiltonian on the zGKNspacetime implies that its naked singularity does no harm.

We have also begun a study of the Dirac Hamiltonian for a zGKN ring singularity exposed to a harmonic magnetic field added into the zGKN spacetime. We plan to report on this in the future.

Another obviously important question is to tackle the problem of two interacting zGKN-like ring singularities, in particular with view toward positronium theory. In that case the Born– Oppenheimer approximation is clearly non-sensical, and one has to face a two-body problem. Here one may hope to benefit from the standard many-body approach detailed in [4].

All these problems should be approachable with the quasi-static approximation invoked here, as long as one is concerned with the discrete spectrum or at most gentle motions.

To go beyond the quasi-static approximation will be a truly challenging task because then one has to contemplate deformations of the ring singularities: stretching, bending, twisting, curling, and who knows what else. It would also require, presumably, solving Einstein's vacuum equations with such dynamical, topologically ring-like singularities as "boundary condition," plus the dynamical Maxwell equations on such a spacetime solution with the right asymptotics. In any event, the generalization of Dirac's equation for this situation should produce the same discrete spectrum as is obtained with the quasi-static approximation.

More speculative, but very tempting and intriguing (to us), are the following thoughts. Namely, since in our interpretation the electron and anti-electron are merely the two different "topo-spin"

states of a single more fundamental particle, quite naturally one may wonder whether in the manybody theory there is an analog of a ferro-magnetic phase transition — this would require a statistical mechanical analysis in the limit of infinitely many zGKN ring singularities. This phase transition would of course not be the magnetization phase transition, but in the topo-spin space, and amount to the emergence of a broken symmetry phase in which only electrons (or only positrons) feature in one sheet of physical space. This would offer a completely novel explanation of the apparently broken particle/anti-particle symmetry in our world!

All of the above invokes the zero-gravity limit. Ultimately one wants to switch gravity back on again, not because one should expect significant corrections to the atomic spectra, but for conceptual reasons. However, as we have recalled, for G > 0 the maximal-analytically extended Kerr–Newman solution suffers from causal pathologies and strong curvature singularities which quite possibly — are unphysical. Upon closer inspection one can trace the origin of these pathologies to the linear Maxwell–Maxwell²² equations which for point and ring sources produce solutions with non-integrable field-energy densities; when coupled to the spacetime structure through the Einstein–Maxwell–Maxwell equations, the consequences are devastating. Thus, before any physically meaningful deformation of the zero-G results to G > 0 can be attempted, first one has to redo the zero-G calculations with the linear Maxwell–Maxwell equations replaced by better-behaved equations that yield integrable energy densities for point and ring charges, for instance the nonlinear Maxwell–Born–Infeld equations, then deform into the full system of Einstein–Maxwell–Born–Infeld equations. So far this has only been accomplished for a single point charge [76] in a topologically simple spacetime, but we don't see any reason for why this should not be possible for spacetimes with Zipoy topology and their ring singularities. Of course, progress may come slowly.

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Appendix

A: From $|\mathbf{q}_{pt} - \mathbf{q}_{rg}|$ and $(\mathbf{q}_{pt} - \mathbf{q}_{rg}) \cdot \mathbf{n}_{rg}$ to oblate spheroidal coordinates

Let $\mathbf{n}_{rg} \in \mathbb{R}^3$ be a unit vector, representing the orientation of the ring, i.e. the ring lies in a plane with unit normal \mathbf{n}_{rg} and is positively oriented with respect to \mathbf{n}_{rg} . Next we introduce oblate spheroidal coordinates (ξ, η, φ) centered at \mathbf{q}_{rg} in order to generate a copy of zGKN with the

²²By speaking of the Maxwell–Maxwell system, the first "Maxwell" stands for the so-called pre-metric Maxwell equations, while the second "Maxwell" stands for the "law of the electromagnetic vacuum" (in condensed matter physics called the "constitutive relations") connecting the four electromagnetic three-component fields of the pre-metric equations. Thus, in Minkowski spacetime, in a space & time splitting of the Faraday tensor **F** into (**E**, **B**) and of the Maxwell tensor **M** into (**D**, **H**), choosing Maxwell's "law of the pure ether" $\mathbf{H} = \mathbf{B}$ and $\mathbf{E} = \mathbf{D}$ yields the Maxwell–Maxwell equations (and the general relativistic model becomes the Einstein–Maxwell–Maxwell equations). A physically potentially important nonlinear law of the electromagnetic vacuum is due to Born and Infeld [13], in which case we shall speak of Maxwell–Born–Infeld, respectively Einstein–Maxwell–Born–Infeld equations; see [52] for a recent discussion of their physical significance. As long as we work with the default choice of Maxwell's "law of the pure ether" we simply drop the second "Maxwell," but retain it in discussions in which it is important to distinguish these from e.g. the Einstein–Maxwell–Born–Infeld equations.

ring singularity of radius |a| centered at \mathbf{q}_{rg} and axis of symmetry parallel to the q^3 axis. These coordinates are defined as follows: let $\mathbf{r} = \mathbf{q} - \mathbf{q}_{rg}$, and define $\xi(\mathbf{r}), \eta(\mathbf{r})$ and $\varphi(\mathbf{r})$ as follows

$$r_1 + ir_2 = a\sqrt{1 + \xi^2}\sqrt{1 - \eta^2}e^{i\varphi}$$
(80)

$$r_3 = a\xi\eta \tag{81}$$

The oblate spheroidal coordinates are closely related to cylindrical coordinates: Let

$$z(\mathbf{q}) := (\mathbf{q} - \mathbf{q}_{\rm rg}) \cdot \mathbf{n}_{\rm rg}$$
(82)

$$\rho(\mathbf{q}) := |(\mathbf{q} - \mathbf{q}_{\rm rg}) \times \mathbf{n}_{\rm rg}| = \sqrt{|\mathbf{q} - \mathbf{q}_{\rm pt}|^2 - z(\mathbf{q})^2}$$
(83)

$$\varphi(\mathbf{q}) := \tan^{-1} \frac{q^2 - q_{\rm rg}^2}{q^1 - q_{\rm rg}^1} + \pi H(-(q^1 - q_{\rm rg}^1)) \operatorname{sgn}(q^2 - q_{\rm rg}^2)$$
(84)

(*H* is the Heaviside function) be cylindrical coordinates centered at \mathbf{q}_{rg} and with symmetry axis parallel to \mathbf{n}_{rg} . We then have the following change of coordinate formula

$$z = a\xi\eta, \qquad \rho = a\sqrt{1+\xi^2}\sqrt{1-\eta^2}, \qquad \varphi = \varphi.$$

In particular, the set of points where $\xi = 0$ is a 2-disc in \mathbb{R}^3 ,

$$\mathcal{D} = \{ \mathbf{q} \in \mathbb{R}^3 \mid z(\mathbf{q}) = 0, \rho(\mathbf{q}) \le a \}$$

and the ring $\mathcal{R} := \partial \mathcal{D}$ is the boundary of this disc. The level sets of ξ are oblate spheroids, and those of η are one-sheeted hyperboloids. As was mentioned before, the coordinate ξ can be extended to take on negative values, and the physical space is likewise extended to become a double-sheeted branched Riemann space [33]. The set of points $(\xi, \eta, \varphi) \in \mathbb{R} \times [-1, 1] \times [0, 2\pi]$ gives us a copy of a spatial slice of the maximal extension of zGKN, which we denote by \mathcal{N} .

Consider now a point particle with location \mathbf{q}_{pt} . The locus of possible ring centers \mathbf{q}_{rg} with ring normal \mathbf{n}_{rg} such that the point particle sits somewhere on that ring, is itself a ring of the same radius, this one around the point particle. Suppose then that we introduce cylindrical coordinates z', ϱ', φ' centered at \mathbf{q}_{pt} with symmetry axis \mathbf{n}_{rg} , in the same manner as in the above. Let \mathbf{q}'_{rg} denote the location of the geometric center of the ring in the primed coordinates based at the point particle. One easily checks that

$$z'(\mathbf{q}_{
m rg}') = -z(\mathbf{q}_{
m pt}), \qquad \varrho'(\mathbf{q}_{
m rg}') = \varrho(\mathbf{q}_{
m pt}), \qquad \varphi'(\mathbf{q}_{
m rg}') = \varphi(\mathbf{q}_{
m pt}) + \pi$$

Let (r', θ', φ') be oblate spheroidal coordinates based at \mathbf{q}_{pt} . Then likewise we have

$$r'(\mathbf{q}_{rg}') = r(\mathbf{q}_{pt}), \qquad \theta'(\mathbf{q}_{rg}') = \pi - \theta(\mathbf{q}_{pt}).$$

It now follows that the interaction potential can be equally well expressed in the primed coordinates:

$$\mathbf{A}_{\mathrm{KN}}^{\mathrm{gen}}(\mathbf{q}_{\mathrm{pt}}) = \mathbf{A}_{\mathrm{KN}}^{\mathrm{gen}\,\prime}(\mathbf{q}_{\mathrm{rg}}^{\prime})$$

Indeed the only difference in the expression for the interaction potential would be that the sign of a is flipped. However, since the geometrical center \mathbf{q}'_{rg} of the ring singularity has no ontological significance, we invoke \mathbf{q}'_{rg} only as auxiliary variable and therefore also not use its oblate spheroidal coordinates based at \mathbf{q}_{pt} , but instead work with the original oblate spheroidal coordinates (r, θ, φ) .

B: Separation of variables for the Dirac Equation on a zGKN spacetime

When $Q = I\pi a$ the Dirac equation (61) for the bi-spinor $\hat{\Psi}$ allows a clear separation also for the remaining r and θ derivatives (commonly referred to in the literature as "radial" and "angular" derivatives, even though r is not a radius and θ is not an angle, except at infinity). Thus, when $Q = I\pi a$ the Dirac equation (61) becomes

$$(\hat{R} + \hat{A})\hat{\Psi} = 0, \tag{85}$$

where

$$\hat{R} := \begin{pmatrix}
imr & 0 & D_{-} - ieQ\frac{r}{\varpi} & 0 \\
0 & imr & 0 & D_{+} - ieQ\frac{r}{\varpi} \\
D_{+} - ieQ\frac{r}{\varpi} & 0 & imr & 0 \\
0 & D_{-} - ieQ\frac{r}{\varpi} & 0 & imr
\end{pmatrix},$$

$$\hat{A} := \begin{pmatrix}
-ma\cos\theta & 0 & 0 & -L_{-} \\
0 & -ma\cos\theta & -L_{+} & 0 \\
0 & L_{-} & ma\cos\theta & 0 \\
L_{+} & 0 & 0 & ma\cos\theta
\end{pmatrix},$$
(86)
$$(86)$$

where D_{\pm} and L_{\pm} have been given in (44). Once a solution $\hat{\Psi}$ to (85) is found, the bi-spinor $\Psi := \mathfrak{D}\hat{\Psi}$ solves the original Dirac equation (21).

Following Chandrasekhar we make the Ansatz that a solution $\hat{\Psi}$ of (85) is of the form

$$\hat{\Psi} = e^{-i(Et - \kappa\varphi)} \begin{pmatrix} R_1 S_1 \\ R_2 S_2 \\ R_2 S_1 \\ R_1 S_2 \end{pmatrix},$$
(88)

with R_k being complex-valued functions of r alone, and S_k real-valued functions of θ alone. Let

$$\vec{R} := \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}, \qquad \vec{S} := \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}.$$
(89)

Plugging the Chandrasekhar Ansatz (88) into (85) one easily finds that there must be $\lambda \in \mathbb{C}$ such that

$$T_{rad}\vec{R} = E\vec{R},\tag{90}$$

$$T_{ang}\vec{S} = \lambda \vec{S},\tag{91}$$

where

$$T_{rad} := \begin{pmatrix} d_{-} & m\frac{r}{\varpi} - i\frac{\lambda}{\varpi} \\ m\frac{r}{\varpi} + i\frac{\lambda}{\varpi} & -d_{+} \end{pmatrix}$$
(92)

$$T_{ang} := \begin{pmatrix} ma\cos\theta & l_{-} \\ l_{+} & ma\cos\theta \end{pmatrix}$$
(93)

The operators d_{\pm} and l_{\pm} are now ordinary differential operators in r and θ respectively, with coefficients that depend on the unknown E, and parameters a, κ , and eQ:

$$d_{\pm} := -i\frac{d}{dr} \pm \frac{-a\kappa + eQr}{\varpi^2}$$
(94)

$$l_{\pm} := \frac{d}{d\theta} \pm (aE\sin\theta - \kappa\csc\theta)$$
(95)

The angular operator T_{ang} in (91) is easily seen to be essentially self-adjoint on $(C_c^{\infty}((0,\pi), \sin\theta d\theta))^2$ and in fact is self-adjoint on its domain inside $(L^2((0,\pi), \sin\theta d\theta))^2$ (e.g. [74, 3]) with purely point spectrum $\lambda = \lambda_n(am, aE, \kappa), n \in \mathbb{Z} \setminus 0$. Thus in particular $\lambda \in \mathbb{R}$. It then follows that the radial operator T_{rad} is also essentially self-adjoint on $(C_c^{\infty}(\mathbb{R}, dr))^2$ and in fact self-adjoint on its domain inside $(L^2(\mathbb{R}, dr))^2$.

Suppose $\vec{R} = (R_1, R_2)^T \in (L^2(\mathbb{R}))^2$ is a non-trivial solution to $T_{rad}\vec{R} = E\vec{R}$, with $E \in \mathbb{R}$. Then

$$\frac{dR_1}{dr} - i\left(E - \frac{a\kappa - eQr}{\varpi^2}\right)R_1 + \frac{1}{\varpi}(imr + \lambda)R_2 = 0$$
$$-\frac{dR_2}{dr} - i\left(E - \frac{a\kappa - eQr}{\varpi^2}\right)R_2 + \frac{1}{\varpi}(imr - \lambda)R_1 = 0.$$

Multiply the first equation by R_1^* and the second equation by R_2^* , add them and take the real part, to obtain

$$\frac{d}{dr}\left(|R_1|^2 - |R_2|^2\right) = 0.$$
(96)

Thus the difference of the moduli squared of R_1 and R_2 is constant, hence zero since they need to be integrable at infinity. I.e.,

$$|R_1| = |R_2| := R. (97)$$

Let $R_j = Re^{i\Phi_j}$ for j = 1, 2. Multiply the first equation by R_2^* , multiply the complex conjugate of the second equation by R_1 , and add them to obtain

$$\frac{d}{dr}\left(\frac{R_1}{R_2^*}\right) = 0. \tag{98}$$

Thus the ratio R_1/R_2^* , and hence the sum of the arguments $\Phi_1 + \Phi_2$ must be a constant, say δ . Thus $R_1 = R_2^* e^{i\delta}$. Since multiplication by a constant phase factor is a gauge transformation for Dirac bi-spinors, we can replace $\hat{\Psi}$ with $\hat{\Psi}' = e^{-i\delta/2}\hat{\Psi}$ without changing anything. The spinor thus obtained has the same form as (88), now with $R'_1 = R'_2^*$. Thus without loss of generality we can assume $\delta = 0$ and $R_1 = R_2^*$.

This motivates us to set

$$R_1 = \frac{1}{\sqrt{2}}(u - iv), \qquad R_2 = \frac{1}{\sqrt{2}}(u + iv)$$
(99)

for real functions u and v. Consider the unitary matrix

$$U := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$
(100)

A change of basis using U brings the radial system (90) into the following standard (Hamiltonian) form

$$(H_{rad} - E) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{101}$$

where

(cf. [81], eq (7.105)) with

$$\gamma := -eQ < 0. \tag{103}$$

Consider now the equations (101) and (91) for unknowns (u, v) and (S_1, S_2) . Let us define new unknowns (R, Ω) and (S, Θ) via the Prüfer transform [68]

$$u = \sqrt{2R}\cos\frac{\Omega}{2}, \quad v = \sqrt{2R}\sin\frac{\Omega}{2}, \quad S_1 = S\cos\frac{\Theta}{2}, \quad S_2 = S\sin\frac{\Theta}{2}.$$
 (104)

Thus

$$R = \frac{1}{2}\sqrt{u^2 + v^2}, \quad \Omega = 2\tan^{-1}\frac{v}{u}, \quad S = \sqrt{S_1^2 + S_2^2}, \quad \Theta = 2\tan^{-1}\frac{S_2}{S_1}.$$
 (105)

As a result, $R_1 = Re^{-i\Omega/2}$ and $R_2 = Re^{i\Omega/2}$. Hence $\hat{\Psi}$ can be re-expressed in terms of the Prüfer variables, thus

$$\hat{\Psi}(t,r,\theta,\varphi) = R(r)S(\theta)e^{-i(Et-\kappa\varphi)} \begin{pmatrix} \cos(\Theta(\theta)/2)e^{-i\Omega(r)/2} \\ \sin(\Theta(\theta)/2)e^{+i\Omega(r)/2} \\ \cos(\Theta(\theta)/2)e^{+i\Omega(r)/2} \\ \sin(\Theta(\theta)/2)e^{-i\Omega(r)/2} \end{pmatrix},$$
(106)

and we obtain the following equations for the new unknowns

$$\frac{d}{dr}\Omega = 2\frac{mr}{\varpi}\cos\Omega + 2\frac{\lambda}{\varpi}\sin\Omega + 2\frac{a\kappa + \gamma r}{\varpi^2} - 2E, \qquad (107)$$

$$\frac{d}{dr}\ln R = \frac{mr}{\varpi}\sin\Omega - \frac{\lambda}{\varpi}\cos\Omega.$$
(108)

Similarly,

$$\frac{d}{d\theta}\Theta = -2ma\cos\theta\cos\Theta + 2\left(aE\sin\theta - \frac{\kappa}{\sin\theta}\right)\sin\Theta + 2\lambda, \tag{109}$$

$$\frac{d}{d\theta}\ln S = -ma\cos\theta\sin\Theta - \left(aE\sin\theta - \frac{\kappa}{\sin\theta}\right)\cos\Theta.$$
(110)

In [53] we show that the above equations, under suitable assumptions on the parameters, have solutions that give rise to an eigenstate $\hat{\Psi} \in \mathsf{H}$.

We conclude by noting that, via a slight change in notation, (106) can be brought to the generalized Cayley-Klein form (17), namely, setting

$$\tilde{R} := \sqrt{2}R(r)S(\theta), \quad \tilde{S} := -Et + \kappa\varphi$$

one obtains

$$\hat{\Psi} = \frac{1}{\sqrt{2}} \tilde{R} e^{i\tilde{S}} \begin{pmatrix} \cos(\Theta/2)e^{-i\Omega/2} \\ \sin(\Theta/2)e^{i\Omega/2} \\ \cos(\Theta/2)e^{i\Omega/2} \\ \sin(\Theta/2)e^{-i\Omega/2} \end{pmatrix} = \frac{1}{\sqrt{2}} \tilde{R} e^{i\tilde{S}} \begin{pmatrix} \check{\psi} \\ \check{\psi}^* \end{pmatrix}$$
(111)

In particular, for bi-spinors of this type, $\Sigma = \pi/2$, so that $\mathbf{v} \cdot \mathbf{n} = 0$ and thus, per discussion in 4.3, the resulting motion of the ring is a stationary circulation.

C: Other proposals to link Dirac's equation and the Kerr–Newman spacetime.

To be sure, we are not the first ones to suspect a connection between the Kerr–Newman (KN) spacetime and the Dirac electron,²³ recall footnote 10. Speculations in this direction seem to have

 $^{^{23}}$ Judged by his early speculations [31], Einstein could have been tempted, too, had he been around in the 60s.

appeared first in Carter's paper [19], where he observed that the Kerr–Newmann solution features a g-factor of 2, just like Dirac's wave equation for the electron. Subsequently this idea was picked up by others, see in particular [67, 59, 2, 17], and related papers mentioned below. However, all these earlier proposals have run into grave difficulties. To some measure these are caused by the physically questionable character of the KN solution with G > 0, unveiled by Carter [19] (see also [42, 65]), while the G = 0 proposals in [67, 59] suffer from artificial infinities introduced by truncating the zGKN manifold to obtain a topologically simple spacetime.

As to its questionable character, the maximal-analytical extension of the axisymmetric and stationary "outer" KN spacetime with G > 0 [19] has a very strong curvature singularity on a timelike cylindrical surface whose cross-section with constant-t hypersurfaces is a circle; here, t is a coordinate pertinent to the asymptotically (at spacelike ∞) timelike Killing field that encodes the stationary character of the "outer regions" of the KN spacetime. This circle is commonly referred to as the "ring" singularity. The region near the ring is especially pathological since it includes closed timelike loops.²⁴ Anybody unfortunate enough to be trapped in this region is doomed to repeat the same mistakes over and over again. In the black-hole sector of the parameter space of the KN family of spacetimes this acausal region is "hidden" behind an event horizon, but the electron parameter values are not in this black-hole sector, the ring singularity is then "naked," and the closed timelike loops turn the entire manifold into a causally vicious set.

As pointed out earlier, it was also discovered in [19] that the ring singularity is associated with the topologically non-trivial feature that the maximal-analytical extension of the KN spacetime consists of two asymptotically flat ends which are "doubly linked through the ring." Carter showed that this Zipoy topology survives the vanishing-charge limit of the KN manifold, which yields the maximal-analytic extension [14] of Kerr's solution [50] to Einstein's vacuum equations, cf. [41]. He furthermore showed that this topology survives the vanishing-mass limit of the Kerr manifold, which yields an otherwise flat vacuum spacetime, which is also the vanishing-mass limit of a static spacetime family discovered and completely described a few years earlier by Zipoy [86].

We emphasize that its nontrivial topology is *not necessarily* a physically questionable feature of the maximal-analytically extended Kerr–Newman solution to the Einstein–Maxwell equations, in the sense that it is not an arbitrary mathematical construct, does not seem to lead to paradoxical conclusions, and no empirical data seem to rule it out yet. Yet it is frequently *perceived* as a physically questionable feature. In particular, some general relativists seem to abhor such topologically non-trivial spacetimes,²⁵ as exemplified by the following quote from [10]: "[Zipoy] endowed [his spacetimes] with rather terrifying topological properties." Guided by similar sentiments, and following [10] in their treatment of the Zipoy spacetimes, Israel [46] engineered a single-leafed electromagnetic spacetime, obtained from half of the maximal-analytically extended KN manifold by identifying the limit points of sequences which approach the disc spanned by the ring from above with those approaching it from below, in *one and the same* leaf. However, this "short-circuiting" procedure produces a spacetime with an *ultra-singular* disc source:²⁶ the disc's interior carries an infinite total charge and current, and with the wrong sign on top of that, which have to be "partly compensated" by an infinite opposite charge and current on its rim to leave a finite net amount of charge and current as diagnosed by the asymptotic flux integrals; see [46], and also [67]. Further-

²⁴The timelike ring singularity of the KN manifold is itself the limit of closed timelike loops.

²⁵Unfortunately this attitude can be encountered even nowadays, despite the ubiquity of topologically non-trivial Calabi–Yau manifolds in string theory.

²⁶Incidentally, for astrophysical interpretations of the Kerr–Newman spacetime (other than a black hole) another cut-and-short-circuiting procedure has been proposed in attempts to find a singular disc source for the spacetime and its electromagnetic fields; see [57, 36], and also [5] for the Kerr spacetime. This results in an infinitely extended, infinitely thin disc source which is not as physically bizarre as the "disc source" spanned by the KN ring singularity.

more, the disc holds a negative infinite amount of mass in its interior, to be "compensated partly" by a positive infinite amount of mass on its rim in order to produce the ADM mass, and it rotates at superluminal speeds [46]. The mathematically artificial construction of a topologically simple spacetime with a "disc source" of such physically bizarre proportions is, in our opinion, too high a price to pay to satisfy — what to us seems to have been — just a prejudice against topologically non-trivial spacetimes,²⁷ expressed in [10, 46]. Lastly, there still is a region of closed timelike loops in Israel's single-leafed G > 0 spacetime.²⁸

By contrast, Arcos and Pereira [2] embrace the topologically non-trivial character of the maximal analytically extended Kerr–Newman spacetime. Yet, also these authors cut and re-glue the G > 0 KN manifold, though in a different way. Namely, in [2] it is argued that the acausal region has to be "cut out" and its boundary "re-glued." Thereby the ring singularity is removed as well, leading to a manifold in the spirit of John Wheeler's "charge without charge" [84]. Unfortunately, as noted in [2], their manifold has the non-orientable topology of a Klein bottle. Furthermore, the authors of [2] only claim the continuity of the metric but do not investigate the higher regularity of the metric and electromagnetic fields across the re-glued cut, and in fact their gluing process may well have introduced some artificial singularities.

Also in contrast to [46, 67, 59], the authors of [2] actually try to establish a mathematical connection between the electromagnetic Kerr-Newman manifold and the Dirac equation for the electron beyond the fact that both feature a g-factor of 2. Interestingly enough, they propose to identify their cut-and-re-glued Kerr-Newman spacetime itself with a bi-spinor solution of Dirac's equation, by mapping the metric components of their manifold into the components of a Dirac bi-spinor which satisfies a Dirac equation for the free electron; a similar proposal was made in [17]. However, such an identification is conceptually unstable under perturbations: first, a Dirac bi-spinor does not have enough degrees of freedom to map into a solution of the Einstein-Maxwell equations in a generic neighborhood of the KN solution; and second, a Dirac bi-spinor satisfying a Dirac equation for a non-free electron will hardly produce a Lorentzian metric (whether using the mapping of [2] or of [17]) that solves the Einstein-Maxwell equations. Thus, the possibility of such an identification for the KN manifold, should it pan out rigorously, would rather seem to be a mathematical curiosity without deeper physical implications.

By considering the zero-G limit of the maximal-analytically extended KN spacetime we are avoiding all the acausal and other pathological features of that spacetime which have prompted the ad-hoc surgical procedures in the studies of [2] and [17] that involve the removal of the ring singularity. In particular, we are able to compute the electromagnetic interaction between the zGKNring singularity and a point charge at rest elsewhere in the zGKN spacetime which provides the interaction term in Dirac's equation — something that is manifestly impossible in the approaches of [2] and [17].

Also others, in particular Lynden-Bell [59] and co-workers [35], have contemplated the zero-G limit of the Kerr–Newman spacetime as a better candidate to provide a link between general relativity and the electron. However, these authors do not consider the maximal-analytical extension but, following Israel's lead [46], introduce an artificial branch cut at the disc spanned

²⁷It is quite an irony of sorts that 35 years earlier Einstein, prejudiced at the time against singularities, together with Rosen [32] envisaged a topologically non-trivial linkage (the so-called Einstein–Rosen bridge) of two copies of an outer region of the Schwarzschild manifold in order to avoid its singularities. As known nowadays, "the Einstein–Rosen bridge cannot be crossed," yet it is part of the maximal-analytical extension of the outer region of the Schwarzschild manifold [56, 75], which — ironically, too — is topologically simple, yet singular, having the past and future, spacelike curvature singularities which Einstein hoped to avoid.

 $^{^{28}}$ To the extent that it has been addressed at all in the pertinent literature, it has only been pointed out that in the zero-charge limit the acausal region is excised by the cutting and short-circuiting process [46], yet in a footnote Israel notes that in the charged situation some acausal region remains after the cutting and short-circuiting.

by the ring. As pointed out in [46], it is possible to interpret the KN electromagnetic fields on Minkowski spacetime, which is the zero-G limit of Israel's single-leafed short-circuited truncation of the maximal-analytically extended KN spacetime. Kaiser [48] finds that interpreted in this way the superluminal speeds are absent, and the rim of the disc rotates exactly at the speed of light; still the interior of the disc carries an infinite total charge and current which are "partly compensated" by an infinite opposite charge and current on the rim of the disc, leaving a finite net amount of charge and current as diagnosed by the usual asymptotic formulas of classical electromagnetism. Undeterred by such warning signs, it has been speculated in [59] "that something similar may occur in the quantum electrodynamic charge distribution surrounding the point electron."

Since nature has her own ways, perhaps she does associate electrons with such ultra-singular discs. However, we have serious doubts that there is any fundamental physical truth in this notion. In particular, from a mathematical point of view the "disc singularity of the Kerr–Newman manifold" is an *artificial construct*, akin to a branch cut, obtained by *arbitrarily*²⁹ choosing the disc spanned by the ring singularity for cutting the maximal-analytically extended Kerr–Newman manifold, then discarding half of it, and short-circuiting the remaining part at the cut.

Besides introducing mathematically artificial pathologies, in Israel's one-leaved truncation of the maximal-analytically extended zero-G Kerr–Newman spacetime the crucial electromagnetic biparticle structure of the zGKN ring singularity is lost, while for us it is the main feature needed to coherently interpret the spectral properties of "Dirac's wave equation for the electron."

Furthermore, as far as we can tell, all previous proposals to link the KN manifold with "Dirac's equation for the electron" invariably invoke Carter's observation that the gyromagnetic ratio of this electromagnetic spacetime (i.e. by definition the ratio of its total magnetic moment to its total angular momentum) is equal to Q/M, which coincides with the value predicted for the electron by the one-body Dirac equation. Yet note that in the zero-G limit the KN spacetime becomes static so that it becomes nonsensical to speak of its gyro-magnetic ratio, indicating that the g-factor may be a false lead. Of course, one may still assign an angular momentum to the ring singularity, but how to do it unambiguously is not clear.

Lastly, to the extend that an estimate of the radius |a| of the KN singularity has been attempted in earlier proposals that try to link the KN manifold with "Dirac's equation for the electron," the suggestive identification of the Kerr–Newman magnetic moment with the Bohr magneton is made, which leads to $2a = \hbar/mc$. In contrast to such estimates, our spectral analysis shows that the Bohr magneton is supplied already by the structure of Dirac's equation for a point particle featuring an electric monopole but no magnetic dipole, so that the magnetic moment carried by the zGKN ring singularity is to be identified with the electron's *anomalous* magnetic moment, yielding a significantly smaller ring radius |a| than in all the other studies.

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²⁹There are uncountably many other ways to cut and then short-circuit one of the so obtained leaves, and each of these artificially-so-created singularities could be claimed to be "the source of the Kerr–Newman fields." One may want to argue that the disc is special because the maximal-analytically extended Kerr–Newman manifold has a reflection symmetry which leaves the disc spanned by the ring singularity invariant, and this is inherited by the "Kerr–Newman manifold with disc singularity;" but that symmetry remains intact also if one cuts along a sphere with the ring singularity as equator, then identifies points in the two hemispheres by reflection.

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