Measuring unrecorded measurement

2016 EPL 115 30005

(http://iopscience.iop.org/0295-5075/115/3/30005)

A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

August 2016

EPL, **115** (2016) 30005 www.epljournal.org
doi: 10.1209/0295-5075/115/30005

Measuring unrecorded measurement

M. REVZEN and A. MANN

Department of Physics, Technion-Israel Institute of Technology - Haifa 32000, Israel

received 27 April 2016; accepted in final form 10 August 2016
published online 6 September 2016

PACS 03.65.Ta – Foundations of quantum mechanics; measurement theory
PACS 03.67.Hk – Quantum communication

Abstract – Projective (von Neumann) measurement of an operator (*i.e.*, a dynamical variable) selected from a prescribed set of operators is termed unrecorded measurement (URM) when both the selected operator and the measurement outcome are unknown, *i.e.*, "lost". Within classical physics a URM is completely inconsequential: the state is unaffected by measurement. Within quantum physics a measurement leaves a mark. The present study provides protocols that allow the retrieval of some of the data lost in a URM. The study is shown as supportive of viewing quantum measurement as made up of both classical-like and pure quantum components.

Introduction. – Within quantum mechanics a measurement (the present study is confined to von Neumann, *i.e.*, projective, measurements [1,2]) has an important effect. This remains true for a measurement of an operator, selected from a prescribed set of operators, with neither the measurement outcome nor the operator chosen being available, *i.e.*, they are "lost". We refer to such measurement as "unrecorded measurement" (URM). *E.g.*, we confine ourselves to a two-dimensional Hilbert space, with measurement of the spin along the x-direction for a particle whose spin is aligned along the y-direction. The measurement outcome has equal chance of being $+1$ or -1. Let the outcome be $+1$. This is a URM measurement if neither the direction (here x) nor the outcome (here $+1$) are available (are "lost"). Nonetheless the state of the system has changed: the particle did enjoy a (von Neumann) measurement. Quantum mechanics (QM) [3] assures us that should we measure the particle's spin along the x-direction, the result would be $+1$ —its pre-measurement notwithstanding. We seek a protocol prescribing the "control measurement" that will disclose (some) of the changes in the system that underwent a URM. The study below considers setups allowing the retrieval of (some of) the unavailable data of a URM. We note that *these are purely quantum in nature as within the classical theory URM is completely inconsequential and leaves, in principle, no trace.*

Quantum measurement (*à la* von Neumann) may logically be considered as made of two components. One essentially quantal and the other of more classical flavour. The quantal part involves the choice of the basis (*i.e.*, the type of measurement undertaken among the prescribed set, *e.g.*, the spin along the x-direction). The system scheduled for measurement is reduced thereby to a "classical-like" distribution. Thus, within our example, the choice of measuring σ_x of some initial state $|\psi\rangle$ is expressed via the reduction of the density matrix to a "classical-like" probability distribution,

$$|\psi\rangle\langle\psi| \rightarrow |+,x\rangle\langle x,+|\psi\rangle\langle\psi|+,x\rangle\langle x,+| \\ + |-,x\rangle\langle x,-|\psi\rangle\langle\psi|-,x\rangle\langle x,-|,$$

with $|\langle +/-,x|\psi\rangle|^2$ being the probability of finding the spin along the x-direction $+1/-1$.

The "classical-like" part involves the actual recorded measurement. Thus, a recorded (= consummated) measurement is viewed as realizing both reductions, that of the density matrix to its "classical-like" (diagonal) distributional form and (its subsequent) projection onto the observed state.

This view of quantum measurement suggests that the step wherein the quantal state is reduced to the "classical-like" distribution with possible outcome within the measured basis is *the* traceable attribute of an unrecorded measurement. It is perhaps natural to consider the state that should allow efficiently the measurement of unrecorded measurement to be a maximally entangled state whose form is basis independent (cf. eq. (13)).

Our strategy is to entangle our system (particle "1" —the system that will undergo a URM) with an ancilla (particle "2") [1]. The URM, though pertaining to particle 1, affects both systems and we shall extract the information we seek from both. This is achieved by a two-particle control measurement of the combined system.

The (single-particle) URM bases that we consider for a d-dimensional Hilbert space particle are mutually unbiased bases (MUB). To assure self-containment and to fix the notation we now give a brief review of MUB [4–14].

Two orthonormal vectorial bases $\mathcal{B}_1$, $\mathcal{B}_2$ are said to be MUB if and only if ($\mathcal{B}_1 \neq \mathcal{B}_2$)

$$\forall\, |u\rangle,\ |v\rangle \in \mathcal{B}_1,\ \mathcal{B}_2, \text{ respectively, } |\langle u|v\rangle| = \frac{1}{\sqrt{d}}. \tag{1}$$

A set of orthonormal bases which are pairwise MUB is a MUB set. It was shown in [11] that there are at most $d+1$ MUB in a set belonging to a d-dimensional space. For $d =$ prime ($d \neq 2$), d members of an MUB set are given in terms of the $(d+1)$-th basis $\{|n\rangle\}, n = 0, 1, \ldots, d-1$ by (b designates a basis, m specifies the vector in the basis)

$$|m;b\rangle = \frac{1}{\sqrt{d}}\sum_{n=0}^{d-1}|n\rangle\omega^{\frac{b}{2}[n(n-1)]-mn};$$
$$b = 0,1,\ldots,d-1;\ \ \omega = e^{i\frac{2\pi}{d}}. \tag{2}$$

The $(d+1)$-th basis, termed the computational basis (CB), is the set of eigenfunctions of the enumerating operator $\hat{Z}$:

$$\hat{Z}|n\rangle = \omega^n|n\rangle. \tag{3}$$

We shall designate this basis with $b = \ddot{0}$; *i.e.*, $|n\rangle = |n;\ddot{0}\rangle$: Thus, the $d+1$ bases are $b = \ddot{0}, 0, 1, \ldots, d-1$.

We adopt the following abbreviation [14]: $|\ddot{m}\rangle = |\ddot{m}; b = \ddot{0}\rangle$ and $|m_0\rangle = |m_0; b = 0\rangle$. We note, for future reference, that the basis $b = 0$ is made of the eigenfunctions of the shift operator $\hat{X}$,

$$\hat{X}|n\rangle = |n+1\rangle;\quad |n+d\rangle = |n\rangle,$$
$$\hat{X}|m;0\rangle = \omega^m|m;0\rangle, \tag{4}$$

i.e., it is the Fourier transform of the CB. Note that the exponents are modular and may be viewed as members of an algebraic field [5–7,15].

Notational convenience leads us to define "tilde states" $|\tilde{m};\tilde{b}\rangle$. These are defined as follows:

$$\langle n|m;b\rangle^* \equiv \langle n|\tilde{m};\tilde{b}\rangle \ \rightarrow\ |\tilde{n}\rangle = |n\rangle;\quad |\tilde{m};\tilde{b}\rangle = |-m;-b\rangle. \tag{5}$$

Note that the variables are modular, *e.g.*, $-m \equiv d - m \pmod d$.

The URM we consider involves measuring an operator $\hat{K}$ of the general form

$$\hat{K}_b = \sum_m |m;b\rangle\omega^m\langle b;m|;\quad b = \ddot{0}, 0, 1, \ldots, d-1, \tag{6}$$

for some selected alignment (= basis), b. The URM considered is a measurement of $\hat{K}_b$, eq. (6), of particle 1 with an outcome m in a basis b with the values of m and b unavailable, lost. Both the initially prepared state and the control measurement basis involve entangled systems: particle 1, the system subjected to the URM, and the ancilla, particle 2.

We shall show below that there are two "natural" control measurements that provide distinct pieces of information. These are measurements of MUB of maximally entangled states (MES) bases. (The MES considered here are pure two-particle states such that partial tracing over either one leaves as unity the density matrix of the other.) The presentation of these MES is simplest with the use of collective coordinates which are now introduced schematically [13–15].

The Hilbert space of two d-dimensional particles, 1 and 2, is spanned by $|n_1\rangle|n_2\rangle,\ n_i = 0, 1, \ldots, d-1,\ i = 1, 2$ where $|n_i\rangle$ is the eigenfunction of $\hat{Z}_i$ ($i = 1, 2$) corresponding to eq. (3), with a similar relation for the shifting operators, $\hat{X}_i$, *viz.* $\hat{X}_i|n_i\rangle = |n_i + 1\rangle$, eq. (4). The space may, alternatively, be accounted for with collective coordinate $|n_c\rangle|n_r\rangle,\ n_j = 0, 1, \ldots, d-1,\ j = c, r$. Here c stands for "center of mass" and r for "relative" coordinates. These are defined for a d-dimensional Hilbert space ($d \neq 2$) via the single-particle dynamical variables (recall, [7,13], that the exponents are modular variables, *e.g.*, $1/2 \equiv (d+1)/2 \pmod d$]), $d =$ odd prime,

$$\hat{Z}_c = \hat{Z}_1^{1/2}\hat{Z}_2^{1/2},\qquad \hat{Z}_r = \hat{Z}_1^{1/2}\hat{Z}_2^{-1/2},$$
$$\hat{Z}_j|n_j\rangle = \omega^{n_j}|n_j\rangle,\qquad \hat{X}_c = \hat{X}_1\hat{X}_2,$$
$$\hat{X}_r = \hat{X}_1\hat{X}_2^{-1},\quad \hat{X}_j|n_i\rangle = |(n+1)_j\rangle,$$
$$\hat{Z}_j\hat{X}_j = \omega\hat{X}_j\hat{Z}_j,\qquad \hat{Z}_i\hat{X}_j = \hat{X}_j\hat{Z}_i,\qquad \hat{X}_j^d = \hat{Z}_j^d = I,$$
$$n_j = 0,1,\ldots,d-1;\ i \neq j;\ i,j = c,r;\ j = c,r. \tag{7}$$

One may consider MUB for the c and r coordinates for the $d+1$ bases. We, however, concern ourselves with the two collective coordinates MUB, the CB, $b = \ddot{0}$, and its Fourier transform, $b = 0$. We label the CB bases of the collective coordinates in close analogy with the particles ones. Thus, the "center of mass", c, CB basis, *i.e.*, the eigenfunctions of $\hat{Z}_c$, are $\{|n_c\rangle\} = \{|n_c;\ddot{0}_c\rangle\}$, $n_c = 0, 1, \ldots, d-1$. With a similar designation scheme for the eigenfunctions of $\hat{X}$, we have the $b = 0$ case. (We shall omit the basis subscript, *e.g.*, $\ddot{0}_c \Rightarrow \ddot{0}$. Whenever possible confusion may arise between values of b for the collective coordinates with those of the single particles, it is removed via a detailed specification.)

Direct calculation proves [13,16] that each $|n_1\rangle|n_2\rangle$ state corresponds to a unique collective state $|n_c\rangle|n_r\rangle$:

$$|n_1\rangle|n_2\rangle \Leftrightarrow |n_c\rangle|n_r\rangle \text{ with}$$
$$n_c = \frac{(n_1+n_2)}{2},\quad n_r = \frac{(n_1-n_2)}{2}$$
$$\Leftrightarrow n_1 = n_c + n_r;\ \ n_2 = n_c - n_r. \tag{8}$$

Adopting the following notational simplification for both c and r, [14], *viz.*

$$|\ddot{n}\rangle_i \equiv |\ddot{n}; b = \ddot{0}\rangle_i,\quad |n_0\rangle_i \equiv |n_0; b = 0\rangle_i,\quad i = c, r,$$

we now prove that the product collective state $|\ddot{m}\rangle_c|2m_0\rangle_r$ is a maximally entangled state (MES). Indeed

$$|\ddot{m}\rangle_c|2m_0\rangle_r = |\ddot{m}\rangle_c\left[\frac{1}{\sqrt{d}}\sum_{n=0}^{d-1}|n\rangle_r\omega^{-2m_0n}\right]$$
$$= \frac{1}{\sqrt{d}}\sum_{n=0}^{d-1}|\ddot{m}+n\rangle_1|\ddot{m}-n\rangle_2\omega^{-2m_0n}, \tag{9}$$

where we used eq. (8). The last expression is obviously a MES, QED.

Now the d^2 MES, $|\ddot{m}\rangle_c|2m_0\rangle_r$, $\ddot{m}, m_0 = 0, 1, \ldots, d-1$, are orthonormal and span the two–d-dimensional–particles Hilbert space and, thus, form a (MES) basis for it. This MES basis defines a conjugate basis made of the d^2 MES: $|\ddot{m}\rangle_r|2m_0\rangle_c$, $\ddot{m}, m_0 = 0, 1, \ldots, d-1$ (r and c are interchanged). The two bases are MUB:

$$|\langle\ddot{m}'|_c\langle 2m'_0|_r|\ddot{m}\rangle_r|2m_0\rangle_c| = \frac{1}{d}, \quad \text{independent of } \ddot{m}, \ddot{m}', m_0, m'_0. \tag{10}$$

Either basis may be used as a retrieving control measurement. Thus, one control measurement involves measuring the operator

$$\hat{\Gamma}^a = \sum_{\ddot{m}, m_0} |\ddot{m}\rangle_c|2m_0; 0\rangle_r \Gamma^a_{\ddot{m}, m_0}\langle\ddot{m}|_c\langle 0; 2m_0|_r. \tag{11}$$

This operator involves the double (commuting) collective operators, eq. (7). (This control measurement relates the URM issue to the so-called Mean King Problem [15,17,18].) The other conjugate control measurement involves measuring

$$\hat{\Gamma}^b = \sum_{\ddot{m}, m_0} |\ddot{m}\rangle_r|2m_0; 0\rangle_c \tilde{\Gamma}^b_{\ddot{m}, m_0}\langle\ddot{m}|_r\langle 0; 2m_0|_c. \tag{12}$$

$\hat{\Gamma}^b$ involves the double (commuting) collective operators conjugate to those of eq. (11) it relates to the so-called Tracking the King problem [15]. In either case $\Gamma_{\ddot{m}, m_0}$ assign arbitrary nondegenerate eigenvalues to the observables.

Measuring unrecorded measurement. – We now outline two protocols wherein measuring a URM allows the retrieval of some of unavailable data of a URM of a d-dimensional particle 1: Let Alice prepare the MES $|0; 0\rangle_c|0; \ddot{0}\rangle_r$, [16,19], wherein particle 1 and an ancilla, particle 2, are (maximally) entangled.

This state is essentially basis independent,

$$|0; 0\rangle_c|0; \ddot{0}\rangle_r = \frac{1}{\sqrt{d}}\sum_n |n\rangle_1|n\rangle_2 = \frac{1}{\sqrt{d}}\sum_m |m; b\rangle_1|\tilde{m}; \tilde{b}\rangle_2. \tag{13}$$

Now let Bob measure $\hat{K}_b$, eq. (6), with an outcome m. (The basis (b) of the measurement and the outcome (m) are not available to Alice. To her the state is an URM state.) The state of the two-particle system is now (unnormalized) [1], using eqs. (2), (8), (9),

$$|m; b\rangle|\tilde{m}; \tilde{b}\rangle. \tag{14}$$

Let Alice select, as her control measurement, to measure $\hat{\Gamma}^b$, eq. (12), having as her outcome, say, $\Gamma^b_{\ddot{m}, m_0}$. Thus, she is assured that the following matrix element is nonvanishing:

$$\langle 2m_0|_c\langle\ddot{m}|_r|m; b\rangle_1|\tilde{m}; \tilde{b}\rangle_2. \tag{15}$$

This implies (note: the equations are modular)

$$\begin{aligned} b &= -\frac{m_0}{\ddot{m}}; \;\; \ddot{m} \neq 0. \\ &= \ddot{0} \;\; \ddot{m} = 0; \; m_0 \neq 0. \\ &\quad \text{undetermined for } \ddot{m} = m_0 = 0. \end{aligned} \tag{16}$$

The last two results are obtained upon evaluating the matrix elements that include $b = \ddot{0}$, *i.e.*, the possibility of Bob measuring $\hat{K}_{(b=\ddot{0})}$.

The protocol above reveals the basis used in the URM except for the case wherein the outcome of the control measurement recovers the initially prepared state which would also be the case wherein no measurement was performed. In such a case the outcome of the control measurement does not reveal any of the sought-after data.

We contend that this is the maximal information that can be extracted on an URM. It reveals the unavoidable disturbance imparted on the measured system involved in the reduction of the quantum state to the "classical like" distribution.

We now wish to consider an alternative kind of information attained via measuring an URM. This is a different sort of of information that is in essence involved in the so-called Mean King Problem.

We now consider once more that Alice prepares the state in eq. (13). Bob measures $\hat{K}_b$ and obtains, say, m, without informing Alice about the basis b and the outcome m. To gain some information of his measurement she measures $\hat{\Gamma}^a$ observing, say, $\ddot{m}, m_0$. Going through similar reasoning as above she now gets

$$\begin{aligned} m &= m_0 + b\ddot{m} - b/2, \;\; b \neq \ddot{0}; \\ &= \ddot{m}, \;\; b = \ddot{0}. \end{aligned} \tag{17}$$

In this case one gets a relation between the basis used (b) in the (unrecorded) measurement and the experimental outcome. This is, perhaps arguably, not as informative as the results above, eq. (16).

Replacing the above with their respective conjugates (*e.g.*, eq. (10)) leads to equivalent results.

Conclusions and remarks. – An Unrecorded Measurement (URM) is a projective (von Neumann) measurement of an operator selected, in our study, from a prescribed complete set of operators with the measurement's outcome and the operator selected unavailable considered as "lost". Such a measurement is completely inconsequential within classical physics (CP). This is, within CP, a measurement that leaves the measured system, in principle, unperturbed. Within quantum mechanics (QM) the measurement marks the measured system. We suggested that the unavoidable mark may be associated with the reduction of the state to "classical-like" distribution that is actuated with the recording of the measurement. The subsequent "loss" of the recording (*i.e.*, both the outcome and the basis chosen) left this as an observable mark.

The present study is confined to d-dimensional Hilbert space particles with d an odd prime as for these dimensionalities the analysis is particularly simple. The extension of the theory to d being a power of prime is possible but is judged to require complicated mathematics without adding physical insight. The case of $d = 2$ requires a special treatment.

The essential role played by entanglement in the unveiling the change of the quantal state due to (projective) measurement relates to an intimate relation among entanglement, measurement theory and the uncertainty principle: the retrieval of data residing in the perturbation of a state due to (projective) measurement requires entanglement. The contention of this work is that the maximal information that can be gained by a single measurement of an unrecorded measurement is the basis used.

REFERENCES

[1] Barnett S. M., *Quantum Information* (Oxford University press, Oxford) 2009.
[2] Nielsen M. A. and Chuang I. L., *Quantum Computation and Quantum Information* (Cambridge University Press) 2000.
[3] Peres A., *Quantum Theory: Concepts and Methods* (Kluwer Academic Publishers, Dordrecht) 1993.
[4] Schwinger J., *Proc. Natl. Acad. Sci. U.S.A.*, **46** (1960) 560.
[5] Wootters W. K., *Ann. Phys. (N.Y.)*, **176** (1987) 1.
[6] Wootters W. K. and Fields B. D., *Ann. Phys. (N.Y.)*, **191** (1989) 363.
[7] Gibbons K. S., Hoffman M. J. and Wootters W. K., *Phys. Rev. A*, **70** (2004) 062101.
[8] Vourdas A., *Rep. Math. Phys.*, **40** (1997) 367; *Rep. Prog. Phys.*, **67** (2004) 267.
[9] Bandyopadhyay S., Boykin P. O., Roychowdhury V. and Vatan F., *Algorithmica*, **34** (2002) 512.
[10] Klimov A. B., Romero J. L., Bjork G. and Sanchez-Soto L. L., *J. Phys. A: Math. Theor. Phys.*, **40** (2007) 3987; *Ann. Phys. (N.Y.)*, **324** (2009) 53.
[11] Ivanovich I. D., *J. Phys. A: Math. Gen.*, **14** (1981) 3241.
[12] Durt T., Englert B.-G., Bengtsson I. and Zyczkowski K., *J. Quantum. Inf.*, **8** (2010) 535.
[13] Revzen M., *Phys. Rev. A*, **81** (2010) 012113.
[14] Revzen M., *J. Phys. A: Math. Theor.*, **46** (2013) 075303.
[15] Revzen M. and Mann A., *Quantum Stud.: Math. Found.*, **3** (2016) 79.
[16] Kalev A., Mann A. and Revzen M., *Phys. Rev. Lett.*, **110** (2013) 260502.
[17] Vaidman L., Aharonov Y. and Albert D., *Phys. Rev. Lett.*, **58** (1987) 1385.
[18] Englert B.-G. and Aharonov Y., *Phys. Lett. A*, **284** (2001) 1.
[19] Kalev A., Mann A. and Revzen M., *EPL*, **104** (2013) 50008.