CONTINUOUS MEASUREMENT AND THE QUANTUM TO CLASSICAL TRANSITION

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SUMMARY

We discuss the quantum to classical transition as a consequence of entanglement of a quantum system with an environment or a detector. Our discussion focuses on the effect of continuous measurement on the system dynamics. We review recent research showing how to model weak continuous measurements as nonlinear dynamical evolutions within the quantum trajectory formalism, and establish a set of necessary quantitative conditions for the quantum to classical transition to take place. This analysis is extended to a bipartite open quantum system, with a spin coupled to a harmonic oscillator as a specific example, to show that classical dynamics emerges despite the persistence of entanglement in the classical limit.

Questions regarding the quantum to classical transition (QCT) have existed since the time of Bohr and Einstein and the inception of quantum mechanics. Although it is widely accepted that a quantum mechanical description holds at the microscopic level of atoms, it was not clear why macroscopic objects that are no more than collections of atoms are described by classical mechanics. Whereas special relativity describes the limit of Newtonian mechanics for objects approaching the speed of light, classical mechanics is not as simply connected to quantum mechanics. In fact, the quantum mechanical Schrödinger equation and Newton's classical equations of motion are quite different entities. They predict very different behaviour, both for the states as well as the dynamics of physical systems. Reconciling the two descriptions has gained new relevance in the emerging field of quantum information science.

The transition from microscopic quantum mechanics to macroscopic classical mechanics is particularly difficult to explain in nonlinear systems that exhibit chaotic classical dynamics. Classical chaos is characterized by exponential sensitivity to initial conditions, quantified by a positive Lyapunov exponent that measures the exponential rate of divergence of neighboring trajectories in phase space [1]. Quantum mechanically, the Heisenberg uncertainty principle does not allow a description of precise trajectories in phase space. Due to the unitary linear evolution of state vectors in quantum mechanics, at first glance there appears to be no corresponding exponential sensitivity to initial conditions for systems described by the Schrödinger equation. Furthermore, even in a semi-classical regime, Einstein pointed out that the general quantization rules that were used to approximate eigenvalues of a quantum system would fail for non-integrable (chaotic) systems due the lack of enough constants of the motion [2]. These arguments led to questions regarding quantum-classical correspondence and the possibility of describing a transition from the quantum to the classical regime.

Here, we review recent studies [3-11] that show how the QCT can be understood both qualitatively and quantitatively by considering *open* quantum systems that are entangled with the environment or a detector. We first present a brief introduction to the role of decoherence in the QCT via entanglement with the environment [3-6] and discuss its limitations for obtaining localized classical trajectories from quantum dynamics. These limitations can be overcome by considering the case where the environment is actually a detector making continuous weak measurements of the quantum system that result in quantum trajectories. We briefly review the theory of quantum trajectories and present an analysis of the QCT via continuous measurements of position. The QCT can be identified by quantifying the amount of localization in phase space and the degree of back-action resulting from the measurement [7]. This approach successfully addresses the question of recovering classical trajectories in regular as well as chaotic systems. Furthermore, the approach can be generalized to the case of bipartite systems in which a single observable is continuously measured. We discuss an illustrative example of a particle in a harmonic trap whose motion is coupled to its spin [9,10]. The effect of continuous measurement of the position of the particle and the subsequent emergence of classical chaos can be simply understood in this system. Quantitative conditions for the QCT can thereby be obtained. Finally, we describe our recent studies of entanglement between spin and motion in this system [11]. These results provide new insight into entanglement of bipartite systems in the classical limit.

DECOHERENCE AND THE QUANTUM TO CLASSICAL TRANSITION

In the early days after the initial development of quantum theory, the problem of the QCT was resolved by resorting

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to an arbitrary division into a quantum world and a classical world. The relation between the two regimes and the location of the division between them remained unclear. A new understanding of the QCT developed with the realization that every quantum mechanical system interacts with its environment, and quantum correlations or *entanglement* can be generated between the system and the environment. A complete description of the system must take into account the effect of this entanglement with the environment. We describe in this section how entanglement with the environment can result in decoherence and yield a classical description of the system dynamics.

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The dynamics of an open quantum system interacting with an unmonitored environment is described by a master equation for its density operator, ρ [12]. The quantum system can equivalently be described in phase space by using a quasi-probability distribution function [13]. In particular, the Wigner function is defined as a partial Fourier transform of the density operator,

$$W(x,p) = \int \mathrm{d}y\, \rho(x-y, x+y) \mathrm{e}^{-2ipy}, \qquad (1)$$

with x and p obeying the usual commutation relation [x,p] = iħ. The Wigner function contains the same information as the density operator. Hence, we can examine the dynamics of the Wigner function rather than the master equation in order to understand the effect of decoherence due to entanglement of the system with the environment. For a standard decoherence model of a quantum system coupled to a bath of harmonic oscillators at finite temperature, evolution of the Wigner function of the quantum system in a potential $V(x)$ can be written as [14]

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \{H,W\}_{\mathrm{PB}} + \sum_n \frac{\hbar^{2n}(-1)^n}{(2n+1)!2^{2n}} \partial_x^{(2n+1)} V \partial_p^{(2n+1)} W + D\partial_p^2 W, \quad (2)$$

where the partial derivatives are with respect to x and p and H refers to the system Hamiltonian alone. This equation is obtained by applying the Wigner transform to the evolution equation for the density operator. We consider the weak coupling regime in which damping is negligible. The first term is equivalent to classical evolution described by the Poisson bracket, the second term describes quantum corrections to the classical evolution, and the last term is a diffusion term resulting from interaction with the environment. As one might expect, the quantum correction terms in Eq. (2) are a function of Planck's constant ħ. Note that for the special case of a quadratic potential, the form of the above equation is that of a classical Fokker-Planck equation in the limit of negligible damping. For a more general nonlinear potential, the dynamics is more interesting, as discussed below.

Consider first, the case where the system is isolated from the environment so that the third term in Eq. (2) is negligible. In order to understand the evolution of the isolated system, we compare the relative strengths of the remaining two terms, the Poisson bracket and the quantum corrections. At first glance it might appear that in a macroscopic regime, where the characteristic actions of the system are much larger than Planck's constant, the quantum correction terms, which are proportional to powers of ħ, should remain small and hence the system should evolve classically according to the Poisson bracket. However, a careful analysis shows that this argument can break down when we deal with classically chaotic systems. In fact, a Hamiltonian chaotic system would exhibit observable nonclassical dynamics on a time scale logarithmic in the size of ħ relative to the characteristic system actions [15,16]. This results from the fact that in a chaotic system, exponential divergence of neighbouring points causes a stretching of the Wigner function in phase space at a ratedetermined by the Lyapunov exponent Λ. The stretching causes the Wigner function to develop coherence over large distances. From Liouville's theorem, the momentum distribution in the conjugate direction to the stretching is also squeezed at an exponential rate. This causes the quantum corrections to the Poisson bracket in Eq. (2) to grow and become non-negligible. At this point the quantum dynamics will diverge from the classical predictions.

Zurek used chaotic orbits of celestial bodies [16] to show that even for macroscopic systems in which ħ is extremely small relative to the characteristic actions of the system, quantum corrections to classical dynamics should be observable after a relatively short amount of time. He estimated this break time to be on the order of a few years for chaotic motion of a moon of Jupiter. This implies that ħ being small is not quite enough to explain the emergence of a classical world from underlying quantum mechanics. Eventually, quantum corrections would become significant and the dynamics would no longer be classical.

A solution to this puzzle was obtained by examining the effect of the diffusion term in Eq. (2). This term is a result of the entanglement of the system with the environment [14]. Decoherence results from this entanglement with the environment and can help to maintain classical evolution [5,6,16]. As described above, in systems with a chaotic classical limit, a break between the predictions of quantum and classical dynamics occurs due to rapid stretching of the chaotic phase space function and exponential squeezing of the momentum. This is because the stretching and squeezing causes the wavefunction to develop coherence over a large extent, so that the evolution can no longer be described by a force that is a local gradient of the potential as in the Poisson bracket term. The diffusion term in Eq. (2) acts to limit the exponential squeezing in the momentum direction and thus diffuses the momentum uncertainty [5]. Momentum diffusion limits the spatial extent over which the wavefunction remains coherent (coherence length). Mathematically speaking, what this means is that the momentum diffusion causes the off-diagonal matrix elements of the density operator $\langle x|\rho|x'\rangle$ to exponentially decay in time. Quantum corrections to classical dynamics generated by the Poisson bracket can be neglected if the density operator has spatial coherence much less than the characteristic distance Δx_{NL} in which the potential is nonlinear, thereby recovering classical dynamics. Decoherence thus ensures classical evolution of the Wigner function in phase space when the diffusion limits the coherence length to be less than Δx_{NL}.

In recent years, a vast amount of theoretical research [17,18] has been performed to understand decoherence in more detail. Furthermore, groundbreaking experiments were conducted to demonstrate and study decoherence in physical systems [19]. New experimental studies are currently underway using a variety of systems including atoms and ions [20], photons [21], superconductors [22], and solid state systems [23]. These studies show that although the initial arguments are generally applicable, every system is different and the specific system parameters and model of the environment can greatly affect the decoherence rate. In particular, in chaotic systems, the effect of decoherence on the QCT is quite complex [24] and further research is required to gain a complete picture of the QCT via decoherence. A deep understanding of decoherence is critical for current applications in quantum information processing in order to design large-scale, error-free quantum information devices.

QUANTUM TRAJECTORIES

Decoherence addresses the question of the QCT for the evolution of phase space distributions [6]. However, it does not address the problem of obtaining localized trajectories from quantum dynamics. A slightly different perspective on decoherence can help to approach this problem. Decoherence can be thought of as a continuous measurement of the system by the environment, with the measurement record being discarded, leading to a corresponding loss of information. If instead, the environment is a detector that is perfectly monitored, the system's state remains pure, evolving according to a stochastic Schrödinger equation that accounts for both Hamiltonian evolution and random evolution conditioned on the measurements. Hence such a description corresponds to an "unravelling of the master equation" [25] describing decoherence, and yields a 'quantum trajectory'. Quantum trajectories are crucial for quantifying the existence of chaos both theoretically and in experiments through the quantitative measure of the Lyapunov exponents. Furthermore, in recent years, as current experiments [26-30] are reaching the point where a quantum system can be continuously monitored, quantum trajectories also provide a way of theoretically studying and interpreting the evolution observed in these experiments .

In order to analyze the QCT via continuous measurements, we must first describe the evolution of the quantum system conditioned on weak measurements. In the theory of projective measurements, a measurement of a system is described by a projection operator acting on the state of the system [31]. However, most physical measurements do not actually correspond to direct projective measurements on the system. Rather, the system of interest is entangled with a detector that subsequently undergoes a projective measurement to gain information about the system. The evolution of an initial product state of the system ρ_S and the meter is ρ_M

$$\rho = U\rho_S(0)\otimes\rho_M(0)U^\dagger. \tag{3}$$

If a measurement of the meter in some basis results in outcome m, then the state of the system after the measurement is

$$\rho_m = \langle m|U\rho_S(0)\otimes\rho_M(0)U^\dagger|m\rangle = \Omega_m\rho_S(0)\Omega_m^\dagger. \tag{4}$$

ρ_m is the unnormalized state of the system after the measurement. The probability for getting outcome m is $P_m = \mathrm{Tr}[\Omega_m^\dagger\Omega_m\rho_S(0)]$. The measurement operators, Ω_m form a Positive Operator Valued Measure (POVM) and satisfy the condition $\sum_m\Omega_m^\dagger\Omega_m = 1$, but are not in general restricted to being projection operators.

If the system were measured but the outcome not known, then the state after the measurement, would be a mixture of all possible final states. Thus, over an infinitesimal time , the non-selective evolution is

$$\rho_S(t+\mathrm{d}t) = \sum_m \Omega_m(\mathrm{d}t)\rho_S(t)\Omega_m^\dagger(\mathrm{d}t). \tag{5}$$

Consider a measurement with two measurement outcomes described by the operators,

$$\begin{aligned}\Omega_1(\mathrm{d}t) &= \sqrt{\mathrm{d}t}\,a,\\ \Omega_0(\mathrm{d}t) &= 1-\left(\frac{\mathrm{i}}{\hbar}H+\frac{1}{2}a^\dagger a\right)\mathrm{d}t.\end{aligned} \tag{6}$$

Substituting these operators into Eq (5), we obtain a master equation in standard Lindblad form [32]

$$\rho_S(t+\mathrm{d}t) = \rho(t) - \frac{\mathrm{i}}{\hbar}[H,\rho(t)]\mathrm{d}t + D[a]\rho(t)\mathrm{d}t, \tag{7}$$

with D being the Lindblad superoperator, $D[a]\rho = a\rho a^\dagger - (a^\dagger a\rho+\rho a^\dagger a)/2$. Thus each Lindblad superoperator in a master equation for Markovian evolution can be described by a measurement process corresponding to a measurement operator. The operators in Eq. (6) for example, can be used in quantum optics to describe photodetection of light emitted from a damped cavity [25]. In this case the operator Ω_1 describes a *jump* occurring when a photon is detected, while Ω_0 corresponds to the evolution in between jumps when no photon is detected. Note that we can unitarily transform the measurement operators in Eq. (6) into a different set of measurement operators corresponding to a different measurement process, without changing the non-selective evolution of Eq. (7). An example of interest is the transformation leading to

$$\Omega'_1(\mathrm{d}t) = (a+\gamma)\sqrt{\mathrm{d}t},$$
$$\Omega'_0(\mathrm{d}t) = 1 - [\frac{\mathrm{i}}{\hbar}H + \frac{1}{2}(\gamma^* a - \gamma a^\dagger) + \frac{1}{2}(a^\dagger + \gamma^*)(a+\gamma)]\mathrm{d}t. \quad (8)$$

This corresponds to the transformation $a \to a+\gamma$ and $H \to H - \frac{i\hbar}{2}(\gamma^* a - \gamma a^\dagger)$. In quantum optics, for $\langle a^\dagger a\rangle << |\gamma|^2$, this describes homodyne detection where the output field is first mixed with a coherent laser field before it is measured. When the outcomes of the measurements are averaged over, these different kinds of measurements will lead to the same master equation, Eq. (7).

Each set of measurement operators thus corresponds to a particular type of measurement and gives a different unravelling of the master equation when the outcomes are retained. A particular unravelling of the master equation for a given set of measurement operators can be described using a stochastic Schrödinger equation. Consider a measurement process described by the measurement operators in Eq. (8). Over a given time interval, $\mathrm{d}t$ either outcome 0 occurs or outcome 1 occurs, depending on the probability P_m of each outcome. At each time step, the quantum state is thus acted on either by Ω'_1 or Ω'_0. The state after the measurement is $\Omega'_m|\psi(t)\rangle/\sqrt{P_m}$ where $m = 0$ or 1. At each step we can randomly pick the operator corresponding to outcome 0 or 1 using the correct probabilities for each outcome. For an initially pure state, this conditioned evolution of the system can thus be described by a stochastic equation [33]

$$|\psi(t+\mathrm{d}t)\rangle = \{1 + \left(\frac{a+\gamma}{\sqrt{\langle(a^\dagger+\gamma^*)(a+\gamma)\rangle}} - 1\right)\mathrm{d}N(t) + \left(\frac{\langle a^\dagger a\rangle}{2} - \frac{a^\dagger a}{2} + \frac{\langle a^\dagger\gamma + \gamma^* a\rangle}{2} - \gamma^* a - \mathrm{i}H\right)\mathrm{d}t\}|\psi(t)\rangle. \quad (9)$$

Here, $\mathrm{d}N(t)$ is a stochastic process that is equal to 1 if outcome 1 occurs or 0 otherwise. When $|\gamma| \to \infty$ such that the number of detections in the time interval $\mathrm{d}t$ is very large, but the evolution of the system is infinitesimal, then the Poisson process $\mathrm{d}N(t)$ can be approximated by a Gaussian or Wiener process, $\mathrm{d}W$ [33], leading to a stochastic Schrödinger equation (SSE) of the form

$$|\psi(t+\mathrm{d}t)\rangle = \{1 - \frac{\mathrm{i}}{\hbar}H\mathrm{d}t - (\frac{1}{2}a^\dagger a - \langle a^\dagger\rangle a + \frac{1}{2}\langle a^\dagger\rangle\langle a\rangle)\mathrm{d}t + (a - \langle a\rangle)\mathrm{d}W(t)\}|\psi(t)\rangle. \quad (10)$$

The SSE thus describes the evolution of a quantum system undergoing continuous weak measurements, where the measurement record at each time step conditions the evolution. Note that if we average over all possible measurement records, then the $\mathrm{d}W$ term disappears and we would recover the non-selective evolution described by the master equation of Eq. (7).

The theory of quantum trajectories presented here has been developed independently by various researchers in different contexts and for numerous purposes over several decades. In particular, quantum trajectories provide a very efficient way to simulate non-selective evolution due to a given master equation, since the master equation dynamics are recovered by averaging over many quantum trajectories. Although not a comprehensive list, we refer the reader to [Refs. 25, 34-40] for more details. Here we describe how this description can be used to analyze and quantify the conditions for the quantum to classical transition. In the following section, we discuss a continuous measurement of position and present a set of inequalities that define the QCT.

THE QUANTUM TO CLASSICAL TRANSITION VIA POSITION MEASUREMENTS

The SSE derived above can be applied to the case of continuous measurements of the position z of a system. The corresponding equation for the density operator $\rho = |\psi\rangle\langle\psi|$ is [25,41]

$$\mathrm{d}\rho = -\frac{\mathrm{i}}{\hbar}[H,\rho]\mathrm{d}t - k[z,[z,\rho]]\mathrm{d}t + \sqrt{2k}(z\rho + \rho z - 2\langle z\rangle\rho)\mathrm{d}W. \quad (11)$$

Here, k is the measurement strength or resolution. We assume perfect measurements such that no information is lost and the overall state remains pure. The evolution of the mean of an observable corresponding to an operator O is given by $\mathrm{d}\langle O\rangle = \mathrm{Tr}[O\mathrm{d}\rho]$. Hence, to order $\mathrm{d}t$, the evolution of the mean position and momentum is

$$\mathrm{d}\langle z\rangle = \frac{\langle p\rangle}{m}\mathrm{d}t + \sqrt{8k}C_{zz}\mathrm{d}W,$$
$$\mathrm{d}\langle p\rangle = \langle F(z)\rangle\mathrm{d}t + \sqrt{8k}C_{zp}\mathrm{d}W \quad (12)$$

where $F(z) = -\partial_z V$ is the derivative of the potential with respect to z, and $C_{ab} = (\langle ab\rangle + \langle ba\rangle)/2 - \langle a\rangle\langle b\rangle$ are the symmetrized covariances. The position measurement results in Gaussian noise (terms proportional to $\mathrm{d}W$) in the evolution of both the mean position and momentum.

One can derive the conditions [7] under which these equations remain close to Newton's equations for the classical variables z and p given by

$$\mathrm{d}z = \frac{p}{m}\mathrm{d}t, \qquad \mathrm{d}p = F(z)\mathrm{d}t. \quad (13)$$

The quantum equations will approach these classical equations when (i) the quantum noise terms proportional to $\mathrm{d}W$ in Eq. (12) remain small and (ii) the state remains localized such that the force term can be expanded around the mean position, $\langle F(z)\rangle = F\langle z\rangle$ + small corrections terms. It follows from the expansion of the force term that the strong localization condition (ii) is satisfied when [8]

$$8k \gg \left|\frac{\partial_x^2 F}{F}\right| \sqrt{\frac{|\partial_x F|}{2m}}. \qquad (14)$$

Since k is the measurement strength, the larger the value of k, the stronger the localization. Condition (ii) thus places a lower bound on k. In the limit of infinite k, the measurement becomes projective. However, this would violate condition (i) of weak disturbance, since the measurement would project the wave function into a position eigenstate. Hence condition (i) puts an upper bound on the value of k. This can be quantified by examining the noise terms proportional to dW, which depend on the covariances C_{ab}. Evolution of the covariances depends on the third moments, which in turn depend on the fourth moments and so on in an infinite hierarchy of equations. When the state remains close to Gaussian, we can truncate the hierarchy at the level of second moments. From the equations of motion for the covariances, one finds that the covariances remain small relative to the total phase space and the weak noise condition is satisfied when [7,8]

$$\frac{2|\partial_x F|}{s} \ll \hbar k \ll \frac{|\partial_x F| s}{4}, \qquad (15)$$

where s is the typical value of the systems action in units of $\hbar$. As s becomes larger the inequality holds for a larger window of k.

The above inequalities provide a quantitative means of determining the QCT. In systems where the characteristic system action is too small relative to $\hbar$ (for example a spin-1/2 system), one cannot find a window of measurement strengths that can satisfy the inequality and the system dynamics typically will not resemble classical dynamics. On the other hand, in the limit of the characteristic actions becoming large relative to $\hbar$ the inequalities can be satisfied and a transition to classical dynamics can be observed. The key point is that the size of $\hbar$ relative to the characteristic actions is not the only parameter in the inequalities and hence is not sufficient by itself to identify the QCT. Indeed, for chaotic systems we have seen, in the Section on Decoherence and the Quantum to Classical Transition, that quantum and classical dynamics can quickly diverge even when $\hbar$ is small relative to the system actions. Equations (14) and (15) show that the strength of the measurement relative to s and $\hbar$ plays an important part in defining the QCT. As shown in [7], even chaotic classical dynamics can be recovered when these inequalities are satisfied. This analysis thus provides a means of quantitatively identifying the 'boundary' between quantum and classical dynamics as well as the process of strong localization together with weak measurement noise by which the QCT occurs.

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THE QUANTUM TO CLASSICAL TRANSITION IN BIPARTITE SYSTEMS

The studies of the QCT presented in the previous section focused on driven systems with one motional degree of freedom, e.g. the Duffing oscillator. This discussion can be extended to bipartite quantum systems. In this section we describe recent studies, performed with our collaborators, of a bipartite system consisting of a spin coupled to a harmonic oscillator [9-11]. When the position of the coupled system is continuously measured, regular as well as chaotic classical dynamics can emerge from the measured quantum trajectories in the limit of large actions of the position and spin relative to $\hbar$. In this limit, the classical Lyapunov exponent can be obtained from the quantum trajectories, providing a quantitative correspondence to the classical chaotic dynamics. Furthermore, the conditions for the QCT can be quantified by placing bounds on the covariance matrix.

A key property of bipartite quantum systems, that was not studied in previous work, is entanglement generated between the coupled subsystems. We can explore the relationship between entanglement and the QCT using the spin-oscillator system. Entanglement is thought to be a highly nonclassical and nonlocal property, and hence we would expect entanglement to be negligible in the classical limit. If the correlation functions between the degrees of freedom of the bipartite system all factorize in the classical limit, then this would ensure that the entanglement disappears. Although this is sufficient, we will show that it is not in fact a necessary condition for the QCT as defined here. Quantum trajectories can closely follow classical dynamics even when there is a large amount of entanglement between the subsystems. These results demonstrate the complexity of defining classical behaviour in coupled systems.

The spin-oscillator system is described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 z^2 + bzJ_z + cJ_x. \qquad (16)$$

This corresponds to a particle of mass m and angular momentum J moving in a harmonic potential with frequency ω. The particle is subject to a constant magnetic field in the x-direction and a position dependent magnetic field in the z-direction. The motion and spin of the particle are coupled via this magnetic field and constitute the two subsystems of interest. This ubiquitous Hamiltonian appears in many areas of physics including

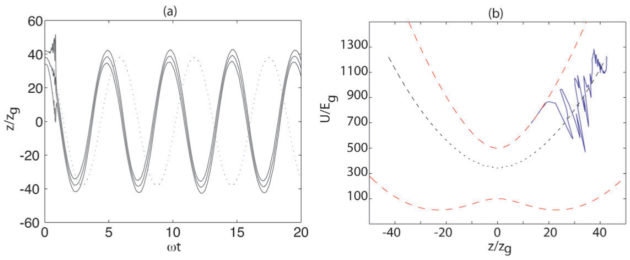

Fig. 1 (a) In the spin 1/2 system the mean position of the measured particle (solid) differs from the classical prediction (dotted). Outer solid curves show the variance of the wave function. See [Ref. 10] for details of the system parameters and initial conditions. (b) The two spinor components of the wave function move along different adiabatic potentials (dashed) until the measurement collapses the wave function into the upper or lower potential. The solid curve shows the resulting measured quantum trajectory. The classical motion is along the dashed-dotted potential. ©American Physical Society

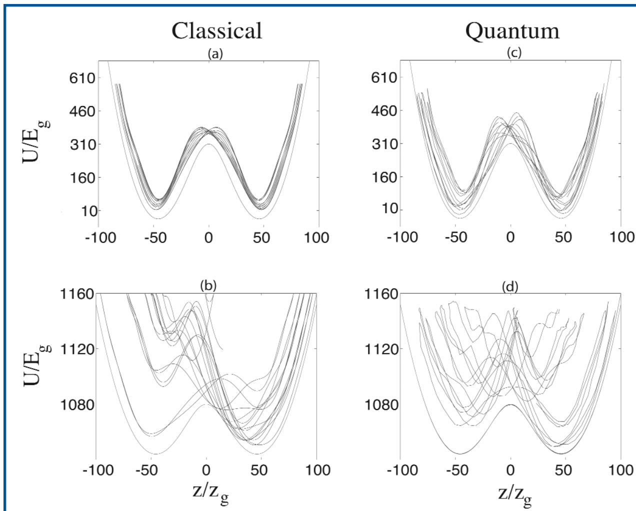

Fig. 2 The mixed classical phase space of regular and chaotic classical trajectories (a,b) are well approximated by the quantum trajectories (c,d) with J=200ħ. For details of the system parameters and initial conditions see [Ref. 10]. ©American Physical Society

condensed matter physics [42], quantum optics [43], and atomic physics [44]. A classical analog corresponds to a particle of mass m and magnetic moment μ moving in the same harmonic potential and subject to the same magnetic field. The classical system is nonintegrable with the strength of the magnetic field in the x-direction being the chaoticity parameter [45,46].

The evolution of the system conditioned on a continuous measurement of the position can be modelled using Eq. (11), with initial states chosen to be a product of minimum uncertainty coherent states of translational motion and spin:

$$|\psi(0)\rangle = |\alpha\rangle\,|\theta,\phi\rangle. \quad (17)$$

The motional coherent states $|\alpha\rangle$ are displacements in phase space of the harmonic oscillator ground state $|0\rangle$ by an amount $\alpha = z + ip$ [47],

$$|\alpha\rangle = e^{\alpha a^{\dagger} + \alpha^{*} a}\,|0\rangle, \quad (18)$$

with a and $a^{\dagger}$ being the creation and annihilation operators. The spin coherent states are rotations of the state $|J, m = J\rangle$ [48],

$$|\theta,\phi\rangle = e^{i\theta\left[J_x \sin\phi - J_y \cos\phi\right]}\,|J, m = J\rangle. \quad (19)$$

System parameters are chosen such that when $b = c = 0$, the motion of the particle in the oscillator alone satisfies the conditions for the QCT defined in the Section on the The Quantum to Classical Transition via Position Measurements. This allows us to study how entanglement with the spin subsystem affects the dynamics. Figure 1 shows the measured evolution of the system when the particle is coupled via the magnetic field to a spin-1/2 subsystem. The quantum trajectories do not agree with the dynamics predicted by classical equations of motion. On the other hand, when the magnitude of the spin subsystem is increased to $J = 200\hbar$, the quantum trajectories approach the classical dynamics in both regular and chaotic regimes (Figure 2).

The advantage of the spin-oscillator system is that we can easily visualize the effect of the measurement on the system and hence exactly understand the process by which the system makes the quantum to classical transition. The dynamics of the spin-1/2 system can be understood by considering the evolution in the basis of adiabatic eigenstates as shown in Fig. 1(b). The two adiabatic potentials and corresponding eigenstates are obtained by diagonalizing the potential at each position. The initial state, which is localized in position, is in a superposition of the two adiabatic eigenstates. The two components of the superposition move along two different potentials (dashed lines in Fig. 1(b)) and hence the wavefunction becomes spatially delocalized into two components. The position measurement acts to localize the wavefunction and hence eventual projects the state into one of the two eigenstate components. At this point the quantum and classical dynamics diverge, since the classical dynamics predicts motion on an average of the quantum adiabatic potentials rather than one or the other potential. The coupling to the quantum spin-1/2 system thus causes the weak position measurement to become a strong projective measurement. Hence the weak measurement condition is violated and classical dynamics is not recovered.

When the magnitude J of the angular momentum subsystem becomes large, the quantum trajectories match the classical trajectories (Fig. 2). For a large spin J, the initial wavefunction is in a superposition of $2J + 1$ adiabatic eigenstates that move $2J + 1$ along potentials. Most of the components move along potentials close to the local direction of the classical magnetic moment and the overall spread of the wavefunction is small. The position measurement acts only to damp the tails of the distribution and does not project the system into a single adiabatic state. Hence the QCT conditions of strong localization and weak backaction can simultaneously be satisfied for both the position and the spin. Furthermore, the classical Lyapunov exponent can be recovered from the quantum trajectories in the large spin limit [10], confirming the quantitative transition to classical chaos.

The conditions for the QCT numerically obtained above can be analytically derived by considering the evolution of the mean position, momentum and spin of the measured quantum system as derived from Eq (11),

$$\begin{aligned} d\langle z\rangle &= \frac{\langle p\rangle}{m}dt + \sqrt{8k}C_{zz}dW, \\ d\langle p\rangle &= -m\omega^2\langle z\rangle dt - b\langle J_z\rangle dt + \sqrt{8k}C_{zp}dW, \\ d\langle \mathbf{J}\rangle &= \gamma\langle \mathbf{J}\rangle\times\mathbf{B}(\langle z\rangle)dt + \gamma C_{\mathbf{J}\times\mathbf{B}(z)}dt + \sqrt{8k}C_{z\mathbf{J}}\,dW, \end{aligned} \quad (20)$$

with $\mathbf{B} = -(c\mathbf{e}_x + bz\mathbf{e}_z)/\gamma$ and the symmetrized covariances C_{ab} defined as $(\langle ab\rangle + \langle ba\rangle)/2 - \langle a\rangle\langle b\rangle$. As described in this section, for systems with one degree of freedom, the measurement must be strong enough to localize the state in phase space so that it resembles a classical point, but weak enough to cause minimal measurement noise or backaction. These two conditions lead to the requirement that the covariance matrix of all the second cumulants must remain small at all times [10]. If the conditioned state remains almost Gaussian in the large action limit, the third and higher cumulants can be neglected and the evolution of the second cumulants can be written in terms of a matrix Ricatti equation,

$$\dot{C}(t) = U + C(t)VC(t) + WC(t) + C(t)W^T, \quad (21)$$

where the matrices U and V depend on the measurement stength and W depends on the system parameters. Numerical studies of the Ricatti equation showed that the covariance matrix remains small relative to the phase space of the motion in the large spin regime where classical trajectories are recovered as expected [10].

ENTANGLEMENT AND THE QUANTUM TO CLASSICAL TRANSITION

We now discuss our recent studies [11] of the behavior of entanglement between spin and motion in the large action limit where classical trajectories are recovered. Since the overall state remains pure, the entropy of either subsystem quantifies the entanglement. For simplicity, we use the linear entropy $S = 1 - \mathrm{Tr}(\rho^2)$ rather than the von Neumann entropy as our measure of entanglement. For the reduced spin subsystem with spin J, the maximum value of S is $S_{max} = 1 - 1/(2J + 1)$.

Figure 3 shows the surprising result that as the system is moved further into the classical regime by making the angular momentum larger, the average normalized entanglement $\langle S\rangle/S_{max}$ between spin and motion increases. This implies that in a regime where classical *dynamics* is obtained from the measured quantum trajectories, the *states* remain highly non-classical with large entanglement.

The behaviour of the entanglement can be simply explained in our test system of spin and motion. The entanglement depends on the overlap between the different spinor components of the total wave function. The system is maximally entangled when there is no overlap between these different components. However, even if the different wavepacket components are spatially distinguishable (i.e., the state is near maximally entangled), a weak measurement may not be able to resolve all the different states and thus 'detect' this entanglement (Fig. 3(b)). Such a measurement may nevertheless be strong enough to satisfy the dual conditions for the QCT. For a constant measurement strength k, as the actions increase, more non-overlapping spinor components are unresolved by the measurement, and hence the steady state entanglement increases. Therefore, one can simultaneously satisfy the QCT conditions (covariance matrix remains bounded), thereby acquiring a trajectory predicted by classical Hamiltonian equations, and yet obtain an evolution that results in a highly entangled quantum state.

The above results can be further understood by examining the SSE in Eq. (11) applied to the evolution of a general bipartite system. From this, the evolution of the reduced state ρ_A of the measured subsystem can be obtained by tracing over the other subsystem:

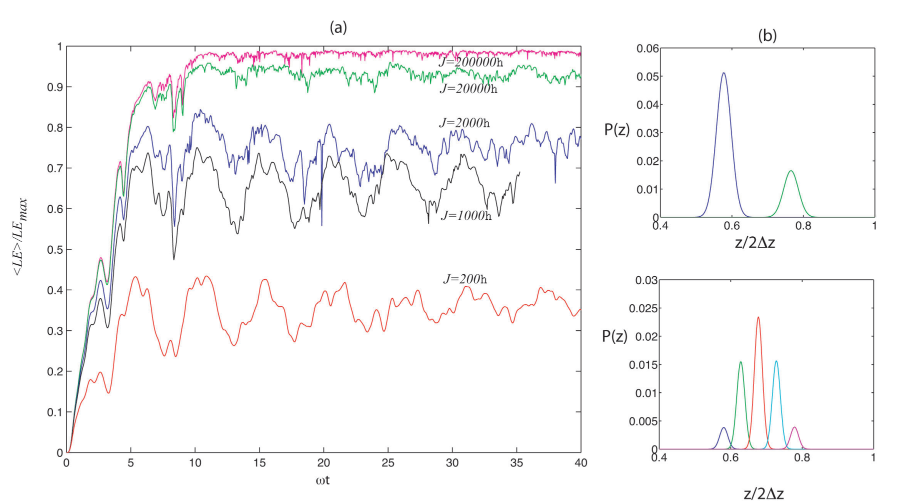

Fig. 3 (a) The normalized average linear entropy of 100 quantum trajectories increases as J is increased. System parameters and initial conditions are given in [Ref. 11]. (b) As the spin increases, more non-overlapping wave packets remain unresolved by the measurement leading to a larger amount of steady state entanglement. ©American Physical Society

$$d\rho_A = -\frac{i}{\hbar}\mathrm{Tr}_B([H,\rho])dt + k(2z\rho_A z - z^2\rho_A - \rho_A z^2)dt \quad (22)$$
$$+\sqrt{2k}(z\rho_A + \rho_A z - 2\langle z\rangle\rho_A)dW.$$

The evolution for the marginal linear entropy S obeys $dS = -2\mathrm{Tr}_A(\rho_A d\rho_A) - \mathrm{Tr}_A[(d\rho_A)^2]$. From Eq. (22) (and only retaining terms to $O(dt)$), the evolution of the entanglement for a given measurement strength k is

$$dS_k = dS_0 - 8k\mathrm{Tr}_A\left[\left(\rho_A(z-\langle z\rangle)\right)^2\right]dt$$
$$-4\sqrt{2k}\mathrm{Tr}_A\left[\rho_A^2(z-\langle z\rangle)\right]dW \quad (23)$$

with dS_0 the term corresponding to measurement-free ($k = 0$) evolution.

In the regime of the QCT, the measurement terms, which can be written in terms of the covariances, must become small, and hence the effect of the measurement on the entanglement decreases. Thus in the classical limit, especially in chaotic systems, the measurement-free evolution term S_0 can become large, with negligible effect from the measurement terms. Hence in the QCT regime in which the condition for the covariances to be small is satisfied, the entanglement can nevertheless be large. The reason that this can occur is because two different scales are involved. The covariances can be small relative to the total phase space of the dynamics, thereby satisfying the conditions for the QCT. On the other hand, assuming the states remain Gaussian in the classical regime, the entanglement is related to the covariances by

$$S = 1 - \frac{\hbar/2}{\sqrt{C_{zz}C_{pp} - C_{zp}^2}} \quad (24)$$

Thus at the same time that the covariances are small relative to the total phase space, if they are large relative to the scale of $\hbar$, then entanglement may be large.

Our analysis of entanglement in continuously measured bipartite systems sheds new light on the quantum classical transition as well as on the standard formalism of von Neumann measurements [49]. It points out that one must be careful about labeling a system as classical when multiple coupled systems are involved. Further studies are required to understand multipartite entanglement and the QCT in more detail.

CONCLUSION

We have come a long way in our understanding of quantum-classical correspondence and the transition from the quantum to the classical world. Here we have presented a summary of our current understanding of the QCT, which is based on the recognition that all systems interact with an environment. When the environment is a detector whose results are monitored, one can obtain quantum trajectories that can be compared to classical trajectories in phase space. Quantitative conditions under which the quantum and classical trajectories will agree can be obtained by using a stochastic Schrödinger equation to describe the measured quantum system and imposing the dual conditions of strong localization and weak measurement noise. This analysis answers the long-standing questions of whether there is a boundary between the quantum and classical world, where this boundary lies and what is the process by which one makes a transition across the boundary.

We have also summarized here recent detailed studies of the QCT in a continuously measured bipartite system of a spin and an oscillator. Classically integrable as well as chaotic motion can be recovered in the measurement record when both the spin and motional actions of the system are large relative to $\hbar$. This generalizes the previous results for systems with a single degree of freedom and shows that in the large spin limit, the two conditions for classicality, namely strong localization and weak measurement noise can be simultaneously satisfied. The conditions for recovering classical dynamics can be quantified by bounding the evolution of the covariance matrix which is described by a matrix Riccati equation.

Our recent studies of the entanglement in continuously measured bipartite system produced intriguing results. We showed that even when there is substantial entanglement between coupled subsystems, classical trajectories can be recovered as long as the measurement is weak and therefore unable to destroy the entanglement. Previous studies of NMR quantum computing have pointed out that the states during a quantum computation can be described classically, but the evolution is difficult to reproduce with classical gates due to signal loss [50]. Our results demonstrate the opposite case - the measured dynamics can be described classically but the states may not. The results highlight the fact that the different notions of classicality of the state vs. the dynamics, are not always compatible. Although we have made significant progress in our understanding of the QCT, the journey is far from over. As more experiments in which good quantum control can be achieved come online, and more theoretical models are developed, we will gain further insights into the surprising world of quantum mechanics and its connection to classical mechanics.

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