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Modeling stock return distributions with a quantum harmonic oscillator

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Abstract – We propose a quantum harmonic oscillator as a model for the market force which draws a stock return from short-run fluctuations to the long-run equilibrium. The stochastic equation governing our model is transformed into a Schrödinger equation, the solution of which features "quantized" eigenfunctions. Consequently, stock returns follow a mixed χ distribution, which describes Gaussian and non-Gaussian features. Analyzing the Financial Times Stock Exchange (FTSE) All Share Index, we demonstrate that our model outperforms traditional stochastic process models, *e.g.*, the geometric Brownian motion and the Heston model, with smaller fitting errors and better goodness-of-fit statistics. In addition, making use of analogy, we provide an economic rationale of the physics concepts such as the eigenstate, eigenenergy, and angular frequency, which sheds light on the relationship between finance and econophysics literature.

Introduction. – In recent years extensive research has been devoted to investigating stock return distributions for asset pricing, risk management, and asset allocation purposes. One important model of stock price evolution is the geometric Brownian motion (GBM), which assumes that the logarithm of a stock price follows a Brownian motion with drift and results in a Gaussian distribution for log stock returns. However, empirical evidence illustrates that the distribution of stock returns has non-Gaussian properties including negative skewness and positive excess kurtosis [1,2]. To describe the characteristics of stock return distribution better, many models have been proposed such as the variance gamma model [3], Laplace distribution model [4], and Heston model [5].

As an alternative to traditional stock return models, an increasing number of quantum models have also been applied to study the stochastic dynamics of stock prices [1,6–16]. Some of these studies successfully capture non-Gaussian properties of the stock return distribution: For instance, Ataullah *et al.* regarded stock returns as a particle evolving in a finite square potential well [1] and Meng *et al.* analyzed the Chinese stock index by means of quantum Brownian motion [16]. The advantage of such quantum models over the traditional stock return models lies in the incorporation of market conditions on the stock returns, which is captured by the potential term in the Hamiltonian. Given these features of quantum models, however, few provide the rationale of choosing potential wells and the economic explanation of physics concepts.

Besides deviations from the Gaussian distribution, another consensus on stock return behavior is that relatively high or low stock returns will dissipate as investors exploit excess profits. This implies that there exists a market force which draws a stock return from short-run fluctuations to long-run equilibrium, which is supported by the evidence of mean reversion in stock returns [17]. To capture this market force, we take, among potentials in quantum models, the harmonic potential; this should give a good

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description since any potential approximates to the harmonic potential near the equilibrium. Specifically, the harmonic potential determines a location-dependent drift term in the stochastic process, with the restoring force proportional to the displacement from the equilibrium. We then consider the Fokker-Planck (FP) equation for the probability density function (PDF) of stock returns, which is converted to a time-independent Schrödinger equation. The well-known mathematical analysis of the Schrödinger equation facilitates an analytic solution for the PDF of stock returns, which features discrete ("quantized") eigenvalues and eigenfunctions. In this paper, we use the term "quantum" to indicate the mathematical description of stock prices, rather than the real quantum nature. Our model outperforms the traditional models, such as the GBM and the Heston model, in fitting the empirical distribution of FTSE All Share Index returns.

There is literature in which derivative pricing was investigated by means of eigenfunction expansion [18,19]. For instance, Davydov and Linetsky unbundled contingent claims into portfolios of primitive securities called eigensecurities. The pricing problem was reduced to a regular Sturm-Liouville problem, the solutions of which form a complete orthonormal basis in the Hilbert space [18]. This is to be contrasted with the physics framework of this paper in which we present the application of the eigenfunction expansion. The Schrödinger equation to which the FP equation is converted takes the form of a Sturm-Liouville equation; on the other hand, the harmonic potential is chosen for a good reason, *i.e.*, as a good approximation near the equilibrium. Furthermore, unlike Davydov and Linetsky who used eigenfunctions simply as a mathematical tool, we provide an interpretation of the eigenspectrum in the context of economics and finance. For example, eigenstates can be regarded as different uncertainty regimes in finance, and the eigenenergy of each state as the degree of investors' collective trading activities, *i.e.*, the pressure on stock prices. The difference in eigenenergies between two states corresponds to the barrier for the stock to overcome and to transmit to higher uncertainty regimes. We also explain the relationship among holding periods, speed of price adjustment, and return volatility in line with finance literature. It is noteworthy that this correspondence between concepts of stock returns and quantum physics has nothing to do with the inherent quantum nature or the interpretational problem in quantum mechanics; however, they provide a heuristic way of thinking about typical questions in the finance area.

This paper consists of five sections: In the second section we propose the quantum-harmonic-oscillator model. The third section presents the methodology and data. In the fourth section we mainly explain economic implications of the physics concept. Finally, the fifth section concludes the paper.

Quantum harmonic oscillator. – Let us consider a standard Wiener process W_t and the following stochastic differential equation:

$$\mathrm{d}x = v(x,t)\mathrm{d}t + \sigma(x,t)\mathrm{d}W_t. \tag{1}$$

Introducing the PDF $\rho(x,t)$ of the random variable x at time t, we obtain the FP equation from eq. (1) [20]:

$$\frac{\partial}{\partial t}\rho(x,t) = \frac{\partial^2}{\partial x^2}\left[D(x,t)\rho(x,t)\right] + \frac{\partial}{\partial x}\left[\rho(x,t)\frac{\partial V(x,t)}{\partial x}\right], \tag{2}$$

where $D(x,t) \equiv \sigma^2(x,t)/2$ is the diffusion coefficient and $V(x,t)$ is the external potential determining the drift term according to $v(x,t) \equiv -\partial V(x,t)/\partial x$. In the simple case of constant D and time-independent potential $V(x)$, eq. (2) can be expressed in terms of the FP operator:

$$\frac{\partial}{\partial t}\rho(x,t) = \left[\frac{\partial^2 V}{\partial x^2} + \frac{\partial V}{\partial x}\frac{\partial}{\partial x} + D\frac{\partial^2}{\partial x^2}\right]\rho(x,t) \equiv \hat{L}\rho(x,t). \tag{3}$$

Note that the operator $\hat{L}$ is non-Hermitian because of the first derivative. This can be remedied by transforming the FP equation in eq. (3) to a Schrödinger equation with a Hermitian Hamiltonian. To achieve this, we introduce a new function [21]:

$$\phi(x,t) \equiv \frac{\rho(x,t)}{\sqrt{\rho_s(x)}}, \tag{4}$$

where $\rho_s(x)$ is the stationary solution of eq. (2) [22]:

$$\rho_s(x) = \frac{1}{C}e^{-V(x)/D} \tag{5}$$

with the normalization constant $C \equiv \int_{-\infty}^{+\infty} \mathrm{d}x\, e^{-V(x)/D}$. Then the FP operator in eq. (3) leads to $\hat{L}\rho(x,t) = -\sqrt{\rho_s(x)}\hat{H}\phi(x,t)$, where the Hermitian Hamiltonian operator $\hat{H}$ is given by

$$\hat{H} = -\frac{1}{2}\frac{\partial^2 V}{\partial x^2} + \frac{1}{4D}\left(\frac{\partial V}{\partial x}\right)^2 - D\frac{\partial^2}{\partial x^2}. \tag{6}$$

The FP equation is now expressed as the time-dependent Schrödinger equation in imaginary time $\tau = -i\hbar t$:

$$i\hbar\frac{\partial}{\partial \tau}\phi(x,\tau) = \hat{H}\phi(x,\tau) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi(x,\tau) + U(x)\phi(x,\tau) \tag{7}$$

with the mass $m \equiv \hbar^2/2D$ and effective potential [23]

$$U(x) \equiv -\frac{1}{2}\frac{\partial^2 V(x)}{\partial x^2} + \frac{1}{4D}\left[\frac{\partial V(x)}{\partial x}\right]^2. \tag{8}$$

The general solution of eq. (7) takes the form

$$\phi(x,\tau) = \sum_{n=0}^{\infty} A_n \phi_n(x) \exp\left(-\frac{i}{\hbar}E_n\tau\right), \tag{9}$$

where A_n is the amplitude of the (normalized) solution $\phi_n(x)$ of the time-independent Schrödinger equation: $\hat{H}\phi_n(x) = E_n\phi_n(x)$ with eigenenergy E_n. The solution of the FP equation thus reads

$$\rho(x,t) = \sqrt{\rho_s(x)} \sum_{n=0}^{\infty} A_n \phi_n(x) \exp(-E_n t), \tag{10}$$

where the amplitude is determined by the initial PDF $\rho(x,0)$ according to

$$A_n = \int_{-\infty}^{\infty} \mathrm{d}x\, \phi_n^*(x)[\rho_s(x)]^{-1/2}\rho(x,0).$$

Note that eq. (7) describes the dynamics of a particle of mass m in the potential $U(x)$. The Taylor expansion of $U(x)$ around the equilibrium point x_0, defined by $\mathrm{d}U/\mathrm{d}x|_{x_0} = 0$, reads

$$U(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{\mathrm{d}^n U}{\mathrm{d}x^n}\right|_{x_0} (x - x_0)^n. \tag{11}$$

In case that deviations from the equilibrium are small, we may neglect terms of higher order in $x - x_0$ and write

$$U(x) = U(0) + \frac{1}{2}kx^2 \tag{12}$$

with $k \equiv \mathrm{d}^2U/\mathrm{d}x^2|_0$, where we have taken $x_0 \equiv 0$ without loss of generality. In this way, $U(x)$ is described by a harmonic potential and the system reduces to a harmonic oscillator.

We thus consider small deviations from the equilibrium and resort to the quantum harmonic oscillator (QHO), which is described by eq. (7) with the effective potential in the form of eq. (12). Specifically, taking the harmonic potential $V(x) = \gamma x^2$, we obtain the effective potential in the harmonic form as well:

$$U(x) = -\gamma + \frac{1}{2}m\omega^2 x^2 \tag{13}$$

with $\gamma = \omega\sqrt{mD/2}$. It is well known that the n-th eigenfunction of the harmonic oscillator is given by

$$\phi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \times \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \tag{14}$$

with the corresponding eigenenergy

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega - \gamma = n\hbar\omega, \tag{15}$$

where H_n is the n-th Hermite polynomial [24].

With eq. (5) given by

$$\rho_s(x) = \sqrt{\frac{m\omega}{\pi\hbar}} \exp\left(-\frac{m\omega}{\hbar}x^2\right),$$

we finally obtain the solution of the FP equation

$$\rho(x,t) = \sum_{n=0}^{\infty} \frac{A_n}{\sqrt{2^n n!}} \sqrt{\frac{m\omega}{\pi\hbar}} \exp(-E_n t) \times H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \exp\left(-\frac{m\omega}{\hbar}x^2\right). \tag{16}$$

Note that this solution takes the form of a mixed χ-distribution:

$$\rho(x,t) = \sum_{n=0}^{\infty} C_n(t)\, \rho_n(x) \tag{17}$$

with $C_n(t) = (A_n/\sqrt{2^n n!})\sqrt{m\omega/\pi\hbar}\, e^{-E_n t}$ and $\rho_n(x) = H_n(\sqrt{m\omega/\hbar}x)e^{-(m\omega/\hbar)x^2}$. For example, we have $\rho_0(x) \propto f(\sqrt{2m\omega/\hbar}x;1)$, $\rho_1(x) \propto f(\sqrt{2m\omega/\hbar}x;2)$, $\rho_2(x) \propto f(\sqrt{2m\omega/\hbar}x;3) - f(\sqrt{2m\omega/\hbar}x;1)$, etc. with $f(x;k) = \frac{2^{1-k/2}}{\Gamma(k/2)}\, x^{k-1}\, e^{-x^2/2}$, where k is the degree of freedom and $\Gamma(z)$ is the Gamma function.

Classically, $F \equiv -\mathrm{d}U/\mathrm{d}x = -kx$ corresponds to the restoring force, which pushes the particle out of the equilibrium position back to the equilibrium one. Further, $\omega \equiv \sqrt{k/m}$ gives the angular frequency of the harmonic oscillator. A higher value of ω leads to faster adjustment to the long-run equilibrium from short-run fluctuations. To understand these physics concepts in the financial context, we may think of x as the deviation of a log stock return from its long-run equilibrium. The mass m can then be regarded as the firm-specific characteristics that determine the speed of price adjustment, such as the market capitalization and trading volumes [11,15,25]. Further, the spring constant k can be considered as common market conditions. This analogy is consistent with the fact that even with the common market force pushing temporarily high or low returns back to the equilibrium, different speeds of price adjustment can be observed across firms [26].

It is of interest that in classical mechanics, the particle position is given by a deterministic function of time t, governed by Newton's law of motion; in the aforementioned financial context, this is analogous to the behavior of stock prices with zero volatility that yields a deterministic trajectory. In reality, however, stock price evolution appears indeed random. Such random fluctuations are mathematically taken into account by quantum fluctuations inherent in the Schrödinger equation formulation and the corresponding probabilistic description is useful in probing the "random evolution" of stock prices.

Since $E_n = n\hbar\omega$, terms of $n \geq 1$ in the summation of eqs. (16) or (17) decays exponentially with time t. In particular, in the limit $t \to \infty$, only the ground-state ($n = 0$) term survives. As a result, the initial memory gets lost and deviation x of log stock returns from the equilibrium follows, regardless of the initial distribution, exactly the Gaussian distribution, in which case the model reduces to the GBM for the price process. At finite time t, on the other hand, the incorporation of excited states ($n \geq 1$)

Table 1: Summary statistics of stock returns for different holding periods (τ).

τ	No. of obs.	Mean	Std.	Skewness	Excess kurtosis
1	1746	0.0508	3.3070	-0.1540	7.0114
5	1742	0.0549	1.3824	-0.6503	5.6287
20	1727	0.0534	0.6545	-1.2350	4.0501

Table 2: Parameter estimates.

Models		τ		
		1	5	20
GBM	μ	0.0704	0.0738	0.0703
	σ^2	0.0433	0.0378	0.0339
Heston	θ	1.558×10^{-4}	1.513×10^{-4}	1.383×10^{-4}
QHO	C_0	0.1708	0.3658	0.7506
	C_1	0.0035	0.0157	0.0646
	C_2	0.0208	0.0299	0.0517
	C_3	-0.0021	-0.0086	-0.0361
	C_4	0.0047	0.0064	0.0096
	$m\omega$	9.666×10^{-36}	4.434×10^{-35}	1.866×10^{-34}

increases the thickness of the tail, displaying leptokurtic. The mixture of even and odd states leads to asymmetry of the distribution, which captures skewness. Therefore the excited states serve to capture the stylized facts of log stock returns, *i.e.*, skewness and kurtosis. Note that the exponential decay takes the form of $\exp(-E_n t)$, thus the inverse value of E_1 serves as a cut-off time of the stock market. When $t > E_1^{-1}$, the contributions of upper levels become much less important. Given a finite time t, the contribution of the n-th excited state decreases exponentially as n is increased. We thus need to consider only a few eigenstates of small n, which makes it feasible to manage eq. (16) for the fitting purpose.

Empirical analysis. – This section demonstrates that the probability distribution derived from our model is fully compatible with empirical data. Specifically, we fit $\rho(x,t)$ in the previous section to the PDF of returns on the FTSE All Shares Index. The fitting procedure consists of two steps: First, we estimate parameters in $\rho(x,t)$ by minimizing an error function, *i.e.*, Cramér-von Mises statistic T_3. Second, we assess the goodness-of-fit with Cramér statistic T_0 to check how well the observed data match the theoretical model with the estimated parameters.

We obtain the daily FTSE All Share Index from the Bloomberg database, restricting the time span from 15 November, 2007 to 21 September, 2014. This period is of interest, as it covers the global financial crisis, the European sovereign debt crisis, and post-recession periods. Different holding periods including day, week, and month are considered for robustness, and then all annualized for consistency of units:

$$x \equiv R_t = \frac{252.5}{\tau} \ln\left(\frac{S_{t+\tau}}{S_t}\right), \tag{18}$$

where τ is the holding period equal to 1, 5, or 20 trading days, and S_t is the closing price of the FTSE All Share Index.

Table 1 summarizes the statistics of stock returns, which are leptokurtic with negative skewness. It is thus manifested that returns do not follow a Gaussian distribution.

We estimate the parameter set Θ that minimizes the distance between the theoretical distribution and the empirical PDF, which is measured by the Cramér-von Mises statistic [1,27]:

$$T_3(\Theta) = \frac{1}{12M} + \sum_{j=1}^{M}\left[F(r_j;\Theta) - \frac{j-1/2}{M}\right]^2, \tag{19}$$

where $r_j \equiv R_j - \bar{R}$ is the j-th ordered centered return with $\bar{R}$ being the historical average return used as a proxy for the long-run equilibrium, M is the total number of observations and $F(r_j;\Theta)$ is the accumulated area under the probability density below the j-th ordered centered return for given parameter set Θ.

As remarked in the previous section, the amplitudes of the 6th and higher eigenstates are rather small and negligible. We thus consider only the first five eigenstates ($0 \le n \le 4$). There are in total six undetermined parameters, which are the amplitudes of five eigenstates, C_n for $n = 0, 1, 2, 3$, and 4, and an additional one $m\omega$ appearing as a whole in the PDF given by eq. (17). We also estimate the parameters of the GBM and the Heston model. Estimated parameters are presented in table 2.

Next we assess the extent to which the models actually reflect the data through the Cramér goodness-of-fit test. The test statistic is of the form

$$T_0 = \sum_{i=1}^{20} \frac{(N_{5i} - E_{5i})^2}{E_{5i}}, \tag{20}$$

Table 3: Cramér goodness-of-fit tests.

τ	GBM	p-value	Heston	p-value	QHO	p-value
1	236.96	0.0000	43.35	0.0007	22.82	0.0633
5	138.05	0.0000	57.95	0.0004	15.63	0.3362
20	186.76	0.0000	148.79	0.0000	20.99	0.1090

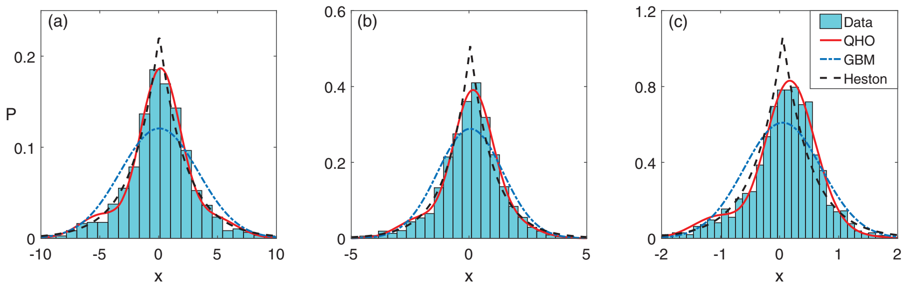

Fig. 1: (Colour online) PDF of log returns x in the GBM (blue), the Heston model (black), and our QHO model (red), for the holding period $\tau =$ 1 (a), 5 (b), and 20 (c). The empirical data are also plotted (histogram).

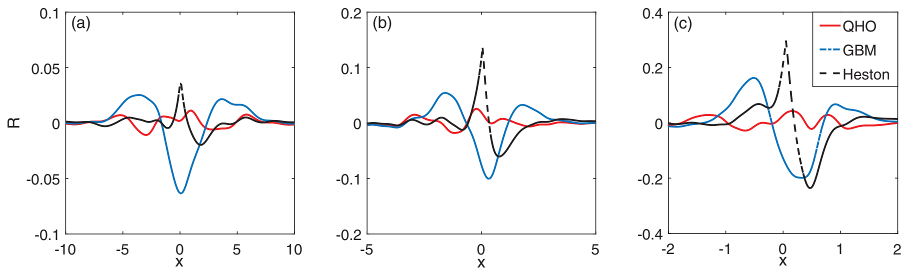

Fig. 2: (Colour online) Residual plots corresponding to fig. 1.

where N_{5i} and E_{5i} are the numbers of observations falling between the $5(i-1)$-th and the $5i$-th percentiles of the empirical PDF and of the fitted distribution, respectively. The asymptotic distribution of T_0 is a $\chi^2(n-k-1)$ distribution, where n is the number of percentiles and k the number of parameters.

The null hypothesis of the Cramér goodness-of-fit test is that data come from the fitted distribution. If T_0 is larger than the critical value, the null hypothesis can be rejected[1]. According to table 3, we reject the null hypothesis for the GBM and the Heston model since all the p-values are smaller than 0.01. In case of the QHO model, the p-value of daily data is larger than 0.05 and the p-values of weekly and monthly data are well above 0.1. Thus one may not reject the null hypothesis that data come from the distribution of the QHO model. In short, the QHO model provides the best fit among the three models.

Discussion. – We plot in fig. 1 the fitted PDF of each model along with the empirical distribution, and in fig. 2 the fitting error of each model. They demonstrate that our QHO model results in the smallest fitting errors, thus visually confirming that our model provides a more adequate description of the empirical distribution. Specifically, the GBM severely understates and overstates the probability density of log returns around zero and in the moderate positive and negative ranges, respectively; the Heston model exaggerates the probability density of small positive or negative returns, and this exaggeration becomes worse as the holding period increases. In contrast, the fitting error of the QHO model remains small in any range of log returns and is affected little by the holding period. Together with the goodness-of-fit statistics shown in table 3, we conclude that the QHO approach outperforms the traditional stock return models.

The sources of such good fit are i) the incorporation of the market uncertainty, which was modeled purely as a random walk in the traditional stock return models, through the properties of wave functions and ii) the consideration of the market force which draws short-run fluctuations to the long-run equilibrium through the QHO.

As addressed in the second section, the solution of the Schrödinger equation is expressed as a linear combination of eigenfunctions corresponding to discrete eigenstates.

[1]Note that the degree of freedom is 17 for the GBM, 18 for the Heston model, and 14 for our QHO model. There are six parameters in the QHO model. However, since they should satisfy one constraint, the number of free parameters reduces to five.

Table 4: Probabilities of five low-lying eigenstates.

τ	P_0	P_1	P_2	P_3	P_4	$m\omega$
1	0.9842	0.0004	0.0145	0.0001	0.0008	9.666×10^{-36}
5	0.9907	0.0018	0.0066	0.0005	0.0003	4.434×10^{-35}
20	0.9856	0.0073	0.0047	0.0023	0.0001	1.866×10^{-34}

Eigenstates are associated with a set of discrete values of physical quantities, such as energy levels. For the n-th eigenstate, the energy E_n and variance σ_n^2 are given by linear functions of the quantum number n: $E_n = n\hbar\omega$ and $\sigma_n^2 = (2n+1)(\hbar/2m\omega)$. The most stable eigenstate is the ground state ($n = 0$), with the lowest energy and variance. In order for a particle to transit to an excited state at a higher energy level, it must absorb energy enough to make a quantum jump to that excited state, which also has a larger variance.

If we regard the variance of the quantum state as the level of market uncertainty, *i.e.*, the stock market volatility, and the energy as the degree of investors' collective trading activities, *i.e.*, the pressure on stock prices, then the QHO model is commensurate with the study in existing finance literature. For instance, it was argued that accumulated price pressure exceeding some threshold can induce large price movements and that the stock market volatility could exhibit a quantum change [28], which is consistent with the properties of the QHO model. In particular, higher market uncertainty corresponds to a higher energy level. Therefore the market tends to limit high volatility by putting a high energy threshold on it.

Note that the probability for the particle in an eigenstate is proportional to the square of the amplitude of that eigenstate. Accordingly, $P_n \equiv N^{-1}|C_n|^2$ with the normalization factor $N \equiv \sum_{k=0}^{4} |C_k|^2$ represents the probability of a stock return residing in the n-th eigenstate. Table 4 presents the probability P_n computed for $n = 0$ to 4. It is shown that the ground state ($n = 0$), following the Gaussian distribution, has a probability higher than 90 percent regardless of the holding period. This indicates that the stock market tends to be bounded mostly at the smallest uncertainty level. In other words, it has a small possibility to be in the eigenstates with higher volatility, which is compatible with the stylized fact that there is an equilibrium level to which volatility will eventually return in the long run [29].

Although the ground state takes the largest probability, we still observe nuances of probabilities across different holding periods. In table 4, as the holding period increases, the probabilities of odd states also increase while those of even states decrease. This makes it possible to explain the properties of moments, shown in table 1: The presence of odd states accounts for the asymmetry of the distribution. A more asymmetric distribution with a larger skewness (longer holding period) would thus have larger probabilities of odd states. On the other hand, even states, which are symmetric, contribute to the fat tail and lead to a higher kurtosis. Therefore we find that returns in longer holding periods have lower probabilities of even states and are less leptokurtic with lower excess kurtosis.

The disparity across different holding periods has its origin in the parameter ω which characterizes the harmonic oscillator. Since m can be thought of as the market capitalization (or firm-specific characteristics in general), it is persistent for different holding periods of one stock index. Therefore, in line with the evidence in table 2, the parameter ω increases as the stock is held for longer period. Since ω is the angular frequency measuring the rate of oscillations around the equilibrium, we can regard ω as the speed of mean reversion of stock returns. During short holding periods when investors aim to speculate in stocks, greater information disparity and the resulting bias lead to price overreaction, thus retarding the price reversion process and leading to a lower speed of mean reversion, and vice versa for long holding periods. On the other hand, a lower mean reversion speed results in a more volatile distribution [30]. This helps to explain the negative relationship between the holding period and stock return volatility, which keeps parallel with Atkins and Dyl [31].

Conclusion. – Considering that the market always draws back the stock return from short-run fluctuations to the long-run equilibrium, we have proposed a model based on a QHO and demonstrated empirical evidence with the FTSE All Share Index. It has been found that our model based on a QHO outperforms the traditional stochastic process models, leading to smaller fitting errors and better goodness-of-fit statistics. The incorporation of market uncertainty through the properties of wave functions is one of the sources of such excellent performance. The model shows that stock returns follow a mixed χ distribution, among which the ground state is Gaussian and the excited states contribute to non-Gaussian features. We also provide the economic analogies of physics concepts: While the eigenstates correspond to uncertainty regimes, the difference in the eigenenergy between two states represents the barrier between the two regimes. It is thus concluded that characteristic features of QHO are indeed relevant although the model has been derived from the stochastic equation for a Wiener process, without regard to quantum mechanics in physics.

One can think of extensions of our approach to various other problems. An example is to apply it to international comparison, *e.g.*, the US *vs.* China, which gives an insight on the difference between the two markets. There exists 10% daily return limitation in the Chinese stock market, in which case the infinite square well might serve as a more proper potential. Another extension involves application to returns of different portfolios, *e.g.*, large *vs.* small, value *vs.* growth, etc. Other than the stock returns, it is also feasible to model the interest rate through the quantum approach and apply it to the bond market. Further, our model can also be applied to risk management, *e.g.*, computing Value at Risk based on the PDF of the quantum harmonic oscillator and comparing it with that from historical simulations or extreme-value theory.

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